Abstract:
In the present paper, we have analyzed nonlinear harmonic response in microwave, photonic devices, and waveguides, applications. Based on Frobenius power series expansions, a special software has been cast and applied with very small truncation error (less than $10^{-8}$). Four basic sets of nonlinear harmonic response systems have been processed: i) New algorithms for computing twelve Jacobian elliptic functions are built on the basis of transformation of Jacobian elliptic integrals to nonlinear differential equations of second order, where each function has its generating recurrence formula., ii) Self-trapping (paraxial propagation) in W-shaped refractive index fibers are parametrically investigated, iii) Implementation of the Frobenius method for the numerical solution of initial value problems involving second order nonlinear differential equations with or without arbitrary forced functions under the forms:

$$y'' + \alpha y' + \beta^2 y^{n/m} = \gamma f(t),$$
$$y'' + \alpha y^{n/m} y + \beta^2 y = \gamma f(t),$$
$$y'' + \alpha y + \beta y^3 = 0,$$

are discussed, where $\alpha$, $\beta$, and $\gamma$ are physical parameters; both $m$ and $n$ are positive solid integers, and $f(t)$ is an arbitrary exciting function, and finally iv) waveguides and multiplexers. A novel software program has been designed and processed to handle the above problems.

I. Introduction
Nonlinear differential equations are usually applied to describe the behavior of many engineering systems. Such systems are nonlinear harmonic response in: microwave and photonics applications, periodic structures, mechanical vibrations, elasticity, fluid mechanics, interaction of radiation and matter propagations problems in communication channels and other applications. Many methods have been applied to solve these equations [1-9]. Most of those solutions were approximate due to the lack of closed form solutions for that type of equations. Power series technique has been successfully applied [2-3] to solve second-order differential equations having nonlinear damping.

In many electronic and mechanical systems, it is important not only to know the frequency of an oscillating system, but also to know how stable the frequency is. Techniques for determining the frequency of oscillation of a given system are plentiful.

Algorithms for computing almost periodic steady state response of nonlinear systems to multiple input frequencies are processed. These algorithms are particularly useful in cases where the steady state response is either not periodic, or is periodic but its period is too large for existing methods.

Current methods for calculating the steady state harmonic response can be classified into four categories:
i) Brute force method via the use of the numerical integration; 
ii) Perturbation method by iteration with the initial solution; 
iii) Harmonic balance method; and 
iv) Shooting method. 
Method (i) is quite general but is expensive for lightly damped circuit. Method (ii) is employed only for almost linear circuits; while method (iii) is extremely time consuming. There are two serious problems associated with method (iii) the first is it can not be used when the solution is not periodic, and the second is it is an expensive numerical technique.

Recently, the response change in linearized systems via computational algorithms which is presented via an investigation of system equations transformed to a parameter-based representation. General matrix modification formulas, applicable to analog and digital systems, are developed for both small and large change response sensitivities with the aid of tremendous new thrust in computer aided design (CAD) and engineering software research and development.

State of the art and present trends in nonlinear microwave CAD technique, as applied to the specific field of microwave circuits, are addressed and developed. A number of fundamental aspects of the nonlinear CAD problem, including simulation, optimization, intermodulation, frequency conversion, stability and noise are reviewed.

Details of the generalized power series technique for the analysis of analog nonlinear systems are reported in Refs.[8, 9] The method uses generalized power series descriptions of the nonlinear element and a spectral balance technique to operate entirely in the frequency domain.

A new algorithm is proposed to calculate the steady state response of a strongly nonlinear system under periodic excitation. The method combines a Volterra approach with some concept adopted in the well-known "harmonic balance" technique.

Computation of the periodic steady state response of nonlinear systems has been carried out with the aid of the extrapolation methods. The performance of the extrapolation methods was demonstrated and compared with other methods for steady state analysis by four examples, two autonomous and two non autonomous. These methods are very easy to implement, and they are efficient for the steady state analysis of nonlinear systems with few reactive elements giving rise to slowly decaying transients.

Kundert and Vincentelli (reported in [8, 9]) reviewed the method of harmonic balance as a general approach to converting a set of differential equations into a nonlinear algebraic system of equations that can be solved for the periodic steady state solution of the original differential equations.

New algorithms for computing twelve Jacobian elliptic functions are processed to calculate and tabulate their values over suitable ranges of interest; these algorithms are built on the basis of transformation of Jacobian elliptic integrals to nonlinear differential equations of second order. Each function is expressed as a series that has its generating recurrence formula. The algorithm handles the generating formula easily through a systematic procedure.

The processed integrals or elliptic Jacobian functions are treated under the usual canonical forms with real parameter less than unity. The twelve treated cases are reduced to general nonlinear differential equation with coefficients depending on the real parameter and each case possesses its own initial conditions.

Despite the increasing use of computers, the basic need for mathematical formulas continues beside the need of mathematical tables. A set of functions of special emphasis in different engineering applications is the set of Jacobian elliptic functions. Such applications are, for examples:

i) Approximation problems in the design of digital devices
ii) Nonlinear Schrodinger wave equation
iii) Solitons in dispersive media, and
iv) The exact solution of the Duffing equation

The processed integrals or elliptic Jacobian functions are treated under the usual canonical forms with real parameter less than unity. The twelve treated cases are reduced to general nonlinear differential equation with coefficients depending on the real parameter and each case possesses its own initial conditions.

The source of the twelve Jacobian elliptic functions is under the normalized canonical form [8,9]
\[ y'' \pm \alpha y' \pm \beta y^3 = 0.0 \quad (1) \]
The ray trajectory [1,10,11] equations in a normalized canonical form has been derived as:
\[ y'' + y - y^3 = 0.0 \quad (2) \]
Mechanical harmonic response have been treated [2,3] on the basis of the following:
\[ y'' + \alpha y' + \beta y^{n/m} = \gamma f(t) \quad (3) \]
and
\[ y'' + \alpha y' + \beta y^{n/m} + \beta y = \gamma f(t) \quad (4) \]
where \( \alpha, \beta, \) and \( \gamma \) are physical parameters.
Propagation in waveguides and multiplexers [10-18], when processed, yields
\[ y'' + \alpha (y^2)^{n/m} + \beta y = 0.0 \quad (5) \]
As the friction term \( \eta y^{n/m} \) and the inertia term \( \omega^2 y^{n/m} \) in the nonlinear differential equations of second order (initial value problem) possess, in general, a fractional number exponent i.e., \( n/m \) is not solid integer.
Thus, in the present paper, we will handle, second order nonlinear differential equations which describe initial value problems of oscillatory features.

II. Basic Model and Analysis
Forced, or unforced, or free, or damped harmonics in engineering systems, in general, obey second order nonlinear differential equations given by:
\[ y'' + \alpha y' + \beta y^{n/m} = \gamma f(t), \quad (6) \]
\[ y'' + \alpha y' + \beta y = \gamma f(t), \quad (7) \]
\[ y'' + \alpha y' + \beta y^3 = 0.0, \quad (8) \]
where Eq. (8) represent in general the ray trajectory and the Jacobian functions [8-10] with special values of the parameters \( \alpha, \beta, \)
where:
\[ m \in \{1,2,3,4,\ldots\}, \quad m \in \{1,2,3,4,\ldots\} \]
\[ \gamma = \begin{cases} \text{1.0 Forced damped oscillations} \\ \text{0.0 Free damped oscillations} \end{cases} \]
with \( y(0) = a_0, \) and \( y'(0) = a_1, \) and \( f(t) = \sum_{j=0}^{\infty} h_j t^j. \)
where \( h_j \) are known set of coefficients.
The starting point in our model is to isolate the nonlinear term as given below:
\[ y^n = (\alpha_0 f(t) - \alpha_1 y - \alpha_2 y^3)^{m} \quad (9) \]
\[ y^n = (\beta_0 f(t) - \beta_1 y^3 - \beta_2 y^3)^{m} \quad (10) \]
Now our suggested model depends on the following series solution:
\[ y(t) = \sum_{j=1}^{m} a_j t^{j-1}, \quad (11) \]
\[ y(t) = \sum_{j=1}^{m} a_j t^{j-1}, \quad (12) \]
and:
\[ y(t) = \sum_{j=1}^{N} j(j+1)a_{j+2} t^{j-2}. \quad (13) \]
where \( a_1, \) and \( a_2 \) are given.
The use of Eqs.(11), (12), and (13) in Eqs.(9) and , (10) yields the following:
A. For Egn.(9):
\[ \left( \sum_{j=1}^{m} j a_{j+1} t^{j-1} \right)^{n} = \left( \sum_{j=1}^{m} j a_{j+1} t^{j-1} \right)^{m} \]
Finally:
\[ \left( \sum_{j=1}^{m} A_{j}^{(1)} t^{j-1} \right)^{n} = \left( \sum_{j=1}^{m} B_{j}^{(1)} t^{j-1} \right)^{m} \]
where the L.H.S is treated as follows:
\[ A_j^{(1)} = ja_{j+1} \quad (16) \]
\[ B_j = \alpha_j h_j - \alpha_j a_j - \alpha_j (j+1)a_{j+2} \quad (17) \]
Now:
\[ \left( \sum_{j=1}^{m} A_{j}^{(2)} t^{j-1} \right)^{2} = \sum_{j=1}^{m} A_{j}^{(1)} t^{j-1} \]
where \( A_j^{(2)} = \sum_{j=1}^{m} A_{j}^{(1)} A_{j-1}^{(1)} \quad (18) \)
Also:
\[ \left( \sum_{j=1}^{m} A_{j}^{(3)} t^{j-1} \right)^{3} = \sum_{j=1}^{m} A_{j}^{(2)} t^{j-1}, \quad (19) \]
and in general:
\[ \left( \sum_{j=1}^{m} A_{j}^{(n)} t^{j-1} \right)^{n} = \sum_{j=1}^{m} A_{j}^{(n)} t^{j-1} \]
where \( A_j^{(n)} = \sum_{j=1}^{m} A_{j}^{(1)} A_{j-1}^{(n-1)} \quad (20) \)
The R.H.S is treated as follows:
Given the following for Eqn.(10):

\[
B_j^{(m)} = \sum_{i=1}^{\lfloor m \rfloor} B_i^{(1)} B_j^{(m-i)}
\]  

(22)

where: \( B_j^{(m)} \) is any solid integer.

We have: \( y(t)^j = \sum_{i=j}^{\lfloor m \rfloor} C_i^{(2)} t^{j-1} \)  

(21)

\[
C_j^{(2)} = \sum_{i=1}^{\lfloor m \rfloor} C_i^{(1)} C_j^{(1)}
\]

(22)

where: \( C_i^{(1)} = a_1 \) and \( C_j^{(1)} = a_2 \)

Thus, we compute:

\[
C_i^{(2)} = C_i^2 = a_1^2, \quad C_j^{(2)} = 2C_i C_j = 2a_1 a_2
\]

Also, the R.H.S. yields:

\[
\sum_{j=1}^{\lfloor m \rfloor} \beta_j t^{j-1} - \beta_j a_j t^{j-1} - \beta_j (j+1) a_j^{j+2}
\]

(23)

\[
= \sum_{j=1}^{\lfloor m \rfloor} D_j^{(3)} t^{j-1}
\]

(24)

\[
D_j^{(1)} = D_1 = \beta_1 a_1 - \beta_2 a_2 - 2a_3^2
\]

(25)

\[
D_j^{(2)} = D_1^{(1)} D_1^{(1)} = D_1^2
\]

(26)

\[
D_j^{(3)} = D_1^{(1)} D_1^{(1)} = D_1^3
\]

(27)

As:

\[
C_i^{(2)} = D_i^{(2)}
\]

we get: \( a_i^2 = (\beta_1 a_1 - \beta_2 a_2 - 2a_3)^2 \)

Thus, both \( a_3 \) and \( D_1 \) are obtained. The process is continued where

\[
C_i^{(2)} = D_i^{(3)} \Rightarrow a_4, D_2
\]

(28)

\[
C_i^{(3)} = D_i^{(3)} \Rightarrow a_5, D_3
\]

(29)

\[
C_i^{(4)} = D_i^{(3)} \Rightarrow a_6, D_4
\]

(30)

\[
D_1^{(1)} = D_1^{(1)} D_1^{(1)} = D_1^2
\]

(31)

\[
D_1^{(3)} = D_1^{(1)} D_1^{(1)} = D_1^3
\]

(32)

C. Case of study 1:

Given the following for Eqn.(10):

\[
\begin{align*}
y_1(0) &= a_1, \\
y_2(0) &= a_2, \\
h_j &= \{h_1, h_2, h_3, h_4, \ldots \}.
\end{align*}
\]

\[
n/m = \begin{cases} n = 2I \\ m = 3I \end{cases}
\]

Given truncation error \( E_t \), the range of solution \( R_t \) is [19-21]:

\[
R_t = \frac{1}{E_t/a_{j+1}}
\]

(33)

D. Case of study 2:

Consider the case of the first Jacobian elliptic integral [8,9], which yields:

\[
y'' + \alpha y' + \beta y^3 = 0 \quad 0
\]

\[
\text{or } y^3 = -A_o y - B_o y'^3
\]

(34)

where \( A_o, B_o \) are in Table I
Table I: The coefficients $A_0$ and $B_0$

<table>
<thead>
<tr>
<th>Function</th>
<th>$A_0$</th>
<th>$B_0$</th>
<th>Initial Values</th>
<th>$y(0)$</th>
<th>$y'(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>sc</td>
<td>-(2-m)</td>
<td>-2(1-m)</td>
<td>0.0</td>
<td>1.0</td>
<td></td>
</tr>
<tr>
<td>cs</td>
<td>-(2-m)</td>
<td>-2</td>
<td>$\infty$ Y(k)=0.0</td>
<td>$\infty$ y(k)=0</td>
<td></td>
</tr>
<tr>
<td>nd</td>
<td>-(2-m)</td>
<td>2(1-m)</td>
<td>1.0</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>dn</td>
<td>-(2-m)</td>
<td>2</td>
<td>1.0</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>sn</td>
<td>1+m</td>
<td>-2m</td>
<td>0.0</td>
<td>1.0</td>
<td></td>
</tr>
<tr>
<td>cd</td>
<td>1+m</td>
<td>-2m</td>
<td>0.0</td>
<td>1.0</td>
<td></td>
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<td>0.0</td>
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<td></td>
</tr>
<tr>
<td>ns</td>
<td>1+m</td>
<td>-2</td>
<td>$\infty$ Y(k)=1.0</td>
<td>$\infty$ y(k)=0.0</td>
<td></td>
</tr>
<tr>
<td>nc</td>
<td>1-2m</td>
<td>-2(1-m)</td>
<td>1.0</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>ds</td>
<td>1-2m</td>
<td>-2</td>
<td>1.0</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>sd</td>
<td>1-2m</td>
<td>-2m(1-m)</td>
<td>0.0</td>
<td>1.0</td>
<td></td>
</tr>
<tr>
<td>cn</td>
<td>1-2m</td>
<td>2m</td>
<td>0.0</td>
<td>1.0</td>
<td></td>
</tr>
</tbody>
</table>

The use of Eqs.(11-13) in Eq.(D-1) gives:

$$
\left[\sum_{j=1}^{\infty} a_j t^{j-1}\right]^3 = \left[ -A_0 \sum_{j=1}^{\infty} a_j t^{j-1} - B_0 \sum_{j=1}^{\infty} j(j+1)a_{j+2} t^{j-1}\right]
$$

$$
= \left[ \sum_{j=1}^{\infty} A_j^{(3)} t^{j-1} = \sum_{j=1}^{\infty} B_j t^{j-1}\right]
$$

Thus: $\sum_{j=1}^{\infty} A_j^{(3)} t^{j-1} = \sum_{j=1}^{\infty} B_j t^{j-1}$

where $B_j = -\left[ A_0 a_j + B_{j0}j(j+1)a_{j+2}\right]$

and $A_j^{(3)} = \sum_{i=1}^{\infty} A_i^{(1)} A_i^{(2)}$

$A_j^{(3)} = \sum_{i=1}^{\infty} A_i^{(1)} A_{j-i+1}^{(2)}$

with $A_j^{(1)} = a_j$.

Finally, from Eq.(D-2) we get:

$A_j^{(3)} = B_j$

(D-3)

Now: $A_j^{(3)} = B_j$

$A_j^{(1)} = a_j$

$A_j^{(2)} = \sum_{i=1}^{\infty} A_i^{(1)} A_i^{(1)} = a_i$

$A_j^{(3)} = \sum_{i=1}^{\infty} A_i^{(1)} A_{j-i+1}^{(1)} = a_i^3$

The use of the above in Eq.(3-D) yields:

$\frac{a_i^3}{2B_{j0}a_j}$ and $a_j$ is computed.

(D-4)

Also: $A_j^{(3)} = A_i^{(1)} A_i^{(2)}$

where $A_i^{(1)} = a_i$

$A_j^{(2)} = \sum_{i=1}^{\infty} A_i^{(1)} A_i^{(1)} = \sum_{i=1}^{\infty} A_i^{(1)} A_i^{(1)}$

$= A_j^{(1)} A_i^{(2)} + A_j^{(2)} A_i^{(1)} = 2a_i a_j$

and

$A_j^{(3)} = \sum_{i=1}^{\infty} A_i^{(1)} A_{j-i+1}^{(1)}$

$= A_j^{(1)} A_i^{(2)} + A_j^{(2)} A_i^{(1)} = 3a_i^2 a_j$

(D-6)

The use of the above in Eq. (D-5) yields:

$A_j^{(3)} = B_j$

i.e., $3a_i^2 a_j = -\left[ A_0 a_j + 6B_{j0}a_j\right]$

(D-7)

Thus $a_j$ is computed, and the process is continued as follows:

$I_j^{(3)} = B_j \Rightarrow a_{j+1}$

(D-7)

where

$a_{j+2} = -\left( A_j^{(3)} - A_j a_j\right)/(j(j+1)B_j)$

(D-8)

III. Conclusions

In the present paper, we have derived novel recurrence relations to obtain the successive coefficients when applying the power series technique to solve nonlinear differential equations of second order which describe initial value problems which possess nonlinear harmonic response in microwave, photonic applications waveguide and multiplexers. Two different equations are treated where the nonlinearity appears respectively in the friction and inertia terms. Two cases of study are processed. A special software program is designed, cast, and applied to process the obtained recurrence relations up to any range according to the required truncation error.

References


