# The Role of Irrational Number Switching Strategies in the Parrondo's Paradox Game

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## ABSTRACT

Parrondo's paradox is a paradox in game theory and is often described as: A losing strategy that wins. According to original formulation of Parrondo's paradox, two separate losing games can be combined to win via a random or periodic strategy. In this paper the role of irrational numbers (e, $\pi$ and Feigenabum constant $\delta$) as a new switching strategy for combination of two losing games on Parrondo's paradox was investigated and the effects of this new strategy on the final gain of Parrondo's paradox was compared to random strategy. Results showed that the highest final gain of Parrondo's paradox is related to the "e" switching strategy.

**Key words**: Irrational number, Parrondo Paradox, Switching strategy, Game theory, Complexity analysis

## 1. INTRODUCTION

Parrondo's game is one of the latest research issues on game theory. It was devised by Parrondo in 1996. After that in 1999 Harmer and Abbott published the first paper on Parrondo's paradox games.[3]

The main idea of Parrondo's paradox is that two individually losing games can be combined to win via periodic or random strategy. There has been a lot of research on Parrondo's games after the first published paper, giving birth to new games such as history dependent games[7] (instead of capital dependent) ,cooperative games[13] (multi-player games instead of mono player).

The original form of the game is shown with biased coins. Game A contains biased coin with the win probability of $\mathbf{p}$ and game B is described as follows:

If the current capital is multiple of $\mathbf{M}$ then the win probability is $\mathbf{p_1}$, otherwise the win probability is $\mathbf{p_2}$.

A convenient parameterization can be introduced if we require to control the three probabilities $\mathbf{p}, \mathbf{p_1}, \mathbf{p_2}$ via a biasing parameter $\boldsymbol{\varepsilon}$. Thus, the parameters are set as the following transformation:$\mathbf{p} = \mathbf{p}' - \boldsymbol{\varepsilon}, \mathbf{p_1} = \mathbf{p}'_1 - \boldsymbol{\varepsilon}, \mathbf{p_2} = \mathbf{p}'_2 - \boldsymbol{\varepsilon}$. The original values of these parameters used in our simulations are: $\mathbf{p}' = \mathbf{1/2}$ ، $\mathbf{p}'_1 = \mathbf{1/10}$، $\mathbf{p}'_2 = \mathbf{3/4}$، $\mathbf{M} = \mathbf{3}$، $\boldsymbol{\varepsilon} = \mathbf{0.005}$.[4] One of the restrictions of this game is that it considered of only two games and one of them is dependent on the capital. In the previous studies complex switching based on random and chaotic strategies have been used to improve the final gain in the classical Parrondo's paradox problem[11]. There are different ways to show either **Game A** or **B** to be played at discrete-time step n, which the simplest one is to compare the chaotic time series $\boldsymbol{x_n}$ to $\boldsymbol{\gamma}$ constant value. If $\boldsymbol{x_n} \geq \boldsymbol{\gamma}$ the **game A** will performed, otherwise, the **game B** will be chosen [14].

In this article, we suggest a new switching strategy based on consequence of transcendental numbers. Irrational numbers can be divided into algebraic and transcendental. A transcendental number is any irrational number that is not a solution of a non-constant polynomial equation with integer coefficients is called algebraic number. Almost all irrational numbers are transcendental, while all transcendental numbers are irrational. The number of decimal digits for any irrational number is infinite. $\boldsymbol{\pi}$ and $\boldsymbol{e}$ are the most prominent examples of transcendental irrational numbers[1]. Besides $\boldsymbol{\pi}$ and $\boldsymbol{e}$, feigenbaum constant $\boldsymbol{\delta}$ is another transcendental irrational number that is widely used in chaos theory [2,12].

To test the ability of this new method, first 200 decimal digits of three irrational numbers $\boldsymbol{\pi}$, $\boldsymbol{e}$ and feigenbaum constant $\boldsymbol{\delta}$ is used as a switching strategy. Finally the simulation results due to these three switching strategies are compared to each other and to random switching strategy to investigate whether these new strategies can improve the final gain in the classical Parrondo paradox problem.

## 1. COMPLEXITY ANALYSIS of TRANSCENDENTAL NUMBERS π, e and FEIGENBAUM CONSTANT δ

The dynamical properties of transcendental numbers $\boldsymbol{\pi}$, $\boldsymbol{e}$, Feigenbaum constant $\boldsymbol{\delta}$ and random numbers can be quantified with fractal dimension via box counting and entropy.

In this study one thousand digits of transcendental numbers, $\boldsymbol{\pi}$, $\boldsymbol{e}$ and Feigenbaum constant $\boldsymbol{\delta}$ have been downloaded from the websites for our analysis and simulation [15]. we constructed time series from the first 200 decimal digits of transcendental numbers $\boldsymbol{\pi}$, $\boldsymbol{e}$, Feigenbaum constant $\boldsymbol{\delta}$ and 200 random numbers by treating the position as the time index and the digit at the position **t**.

There are many methods used to define the fractal dimension of a time series in Euclidean space $\mathbf{R}^{\mathbf{n}}$. Box-counting dimension, is one of the most popular ones[8]. The box-counting dimension $D_b$ is defined to be the number that satisfies

$$D_b = -\lim_{R\to 0}\left\{\frac{\log\ N(R)}{\log\ R}\right\} \qquad (1)$$

Where N(R) is the number of boxes of side length **R** required to cover the set. The estimation of fractal dimensions via Box counting algorithm has been explained theoretically [8,9],[12].

In this study the graph of time series considered as a subsets in the two-dimensional Euclidean space $\mathbf{R}^{\mathbf{2}}$ and the fractal dimensions was calculated. Another mathematical tools for analysis of time series is entropy.

Entropy is a useful method to a description of the state space behavior of a dynamical system and order of non-periodic time series. Therefore in our study the entropy was used to compare the degree of regularity of the three transcendental numbers $\boldsymbol{\pi}$, $\boldsymbol{e}$, Feigenbaum constant $\boldsymbol{\delta}$ with random time series. The higher the entropy of a time series the greater the disorder [5,6].

Entropy estimation is a two stage process; first a histogram is estimated and then the entropy is calculated. The following equation is used to compute the entropy of dynamic system:

$$S = -k\sum_i P_i \ln P_i \qquad (2)$$

Where $\boldsymbol{S}$ is the conventional symbol for entropy. The constant of proportionality $\boldsymbol{k}$ depends on what units are chosen to measure $\boldsymbol{S}$. When SI units are chosen, we have $\boldsymbol{k}$ = kB = Boltzmann's constant = 1.38066×10−23 J K−1. The sum runs over all microstates consistent with the given macrostate and $P_i$ is the probability of the i th microstate introduced by:

$$P_i = \mu(x_i) = \int_{\text{microstate } i} P(x)dx \qquad (3)$$

The mathematical theory of entropy is presented by Martin and James [10]. In our study the programs of computing fractal dimension and entropy was written in Matlab software.Results in Table1 show a significant difference between fractal dimension and entropy of three transcendental numbers $\boldsymbol{\pi}$, $\boldsymbol{e}$, Feigenbaum constant $\boldsymbol{\delta}$ time series and random sequence.

**Table1.** Entropy and fractal dimension of transcendental numbers $\pi$, $e$, feigenbaum constant $\delta$ time series and random sequence.

| **Number** | **Entropy** | **Fractal dimension** |
|---|---|---|
| $\pi$ | 1.0483 | 0.9568 |
| $e$ | 1.0481 | 0.9546 |
| Feigenbaum constant $\delta$ | 1.0490 | 0.9560 |
| Random | 1.2221 | 0.9930 |

In order to specify other differences between transcendental numbers $\pi$, $e$, Feigenbaum constant $\delta$ time series and random sequence the histograms of random and transcendental numbers are plotted. Fig.2
indicates an almost uniform distribution for all time series except for the Random time series.

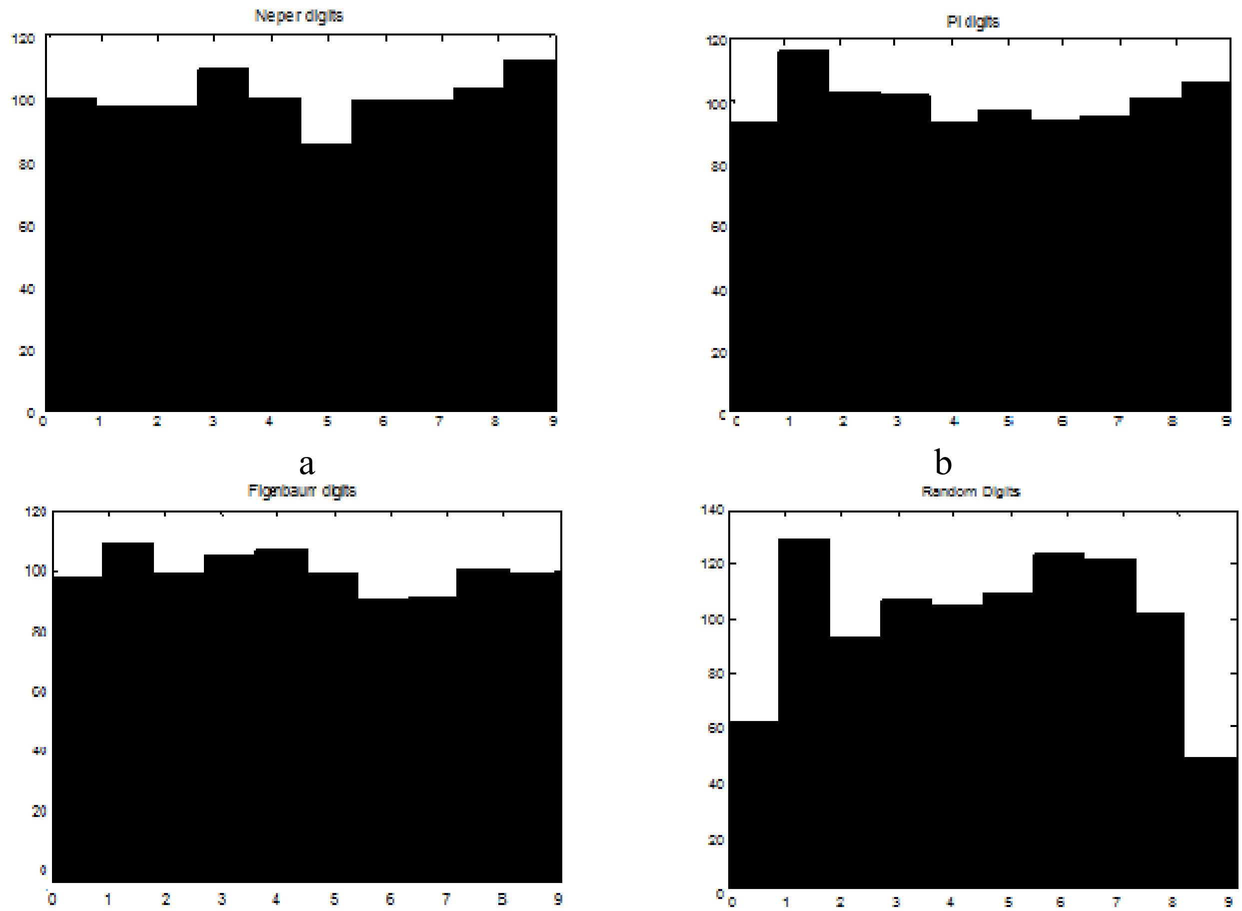


**Figure2.** The histograms of 1000 decimal digits of the transcendental numbers $e$, $\pi$ Feigenbaum constant $\delta$ and random number.
a) $e$ digits time series b) $\pi$ digits time series c)Feigenbaum constant $\delta$ digits time series d)Random digits

Considering the above analysis, we can conclude that the dynamical characteristics of random numbers and transcendental irrational numbers are different.
To see the practical differences between $\pi$, $e$, Feigenbaum constant $\delta$ and random number, in the next section, these four numbers were used as a switching strategy in the classical parrondo's paradox game and their effects on the parrando's paradox final gain were compared.

## 2. $\pi$, $e$ and FEIGENBAUM CONSTANT $\delta$ SWITCHING STRATEGIES

As previously explained only one of the games A or B can be played at any moment. Sequence $x_n$ determines either Game A or B to be played at discrete-time step n. The easiest way for carrying out this task, is to compare each value of $x_n$ to a constant $\gamma$. On each round of Parrondo's game, a value from the time series, $x_n$ is compared to $\gamma$, if $x_n \leq \gamma$., Game A will be played but if $x_n > \gamma$, Game B will be played. On the other hand the $\gamma$ value is acting as a threshold value on deciding whether the next game played should be Game A or B.

In the original Parrondo's game, the switching strategies are random and periodic. In our study the Parrondo's game was simulated using $\boldsymbol{\pi}$, $\boldsymbol{e}$ and Feigenbaum constant $\boldsymbol{\delta}$ strategies to determine the effect of switching strategy type and $\gamma$ parameter values on the final capital of Parrondo's game. In the simulation, each game was played individually 200 times and the outcomes were averaged over 50,000 trials. In Figs. 3 we show how gain changes across iterations in $\boldsymbol{e}$ using some different values of $\gamma$ parameter. Other plots are available upon request.

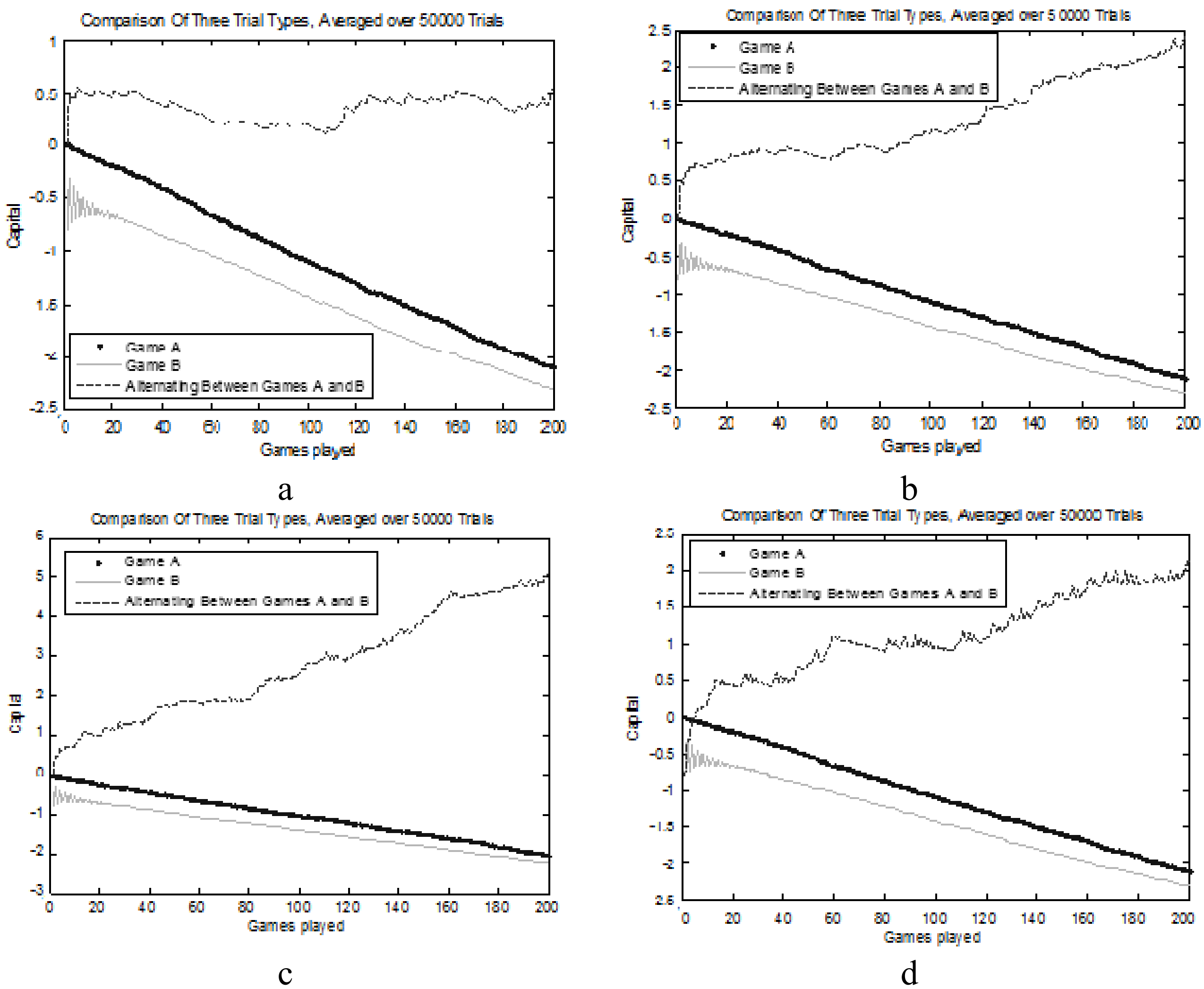


**Fig. 3.**Gain variation with different $\boldsymbol{\gamma}$ values and investigate the effect of this parameter in evolutionary game with $\boldsymbol{e}$ strategy. a- $\boldsymbol{\gamma = 1}$ ,b- $\boldsymbol{\gamma = 2}$, c- $\boldsymbol{\gamma = 5}$,d- $\boldsymbol{\gamma = 7}$ . Each point was obtained as the average over 50,000 trials.

The effects of all values of $\gamma$ on the final gain of Parrondo's paradox game are given in Table 2. It should be noticed that except for random switching strategy, $\gamma = 5$ causes the maximum final gain in all strategies. Also, by selecting the e switching strategy, Parrondo's paradox has a highest final gain in $\gamma = 5$ compared to another three strategies. These results are shown graphically in Fig.5 .

**Table2.** Parrondo's paradox final gain for different values of $\gamma$ constant after 200 iterations

| Strategy | Capital | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $\gamma = 1$ | $\gamma = 2$ | $\gamma = 3$ | $\gamma = 4$ | $\gamma = 5$ | $\gamma = 6$ | $\gamma = 7$ | $\gamma = 8$ |
| $\pi$ | -0.6208 | 0.5588 | 1.1103 | 2.7787 | 3.7883 | 3.4866 | 2.6790 | 0.7171 |
| $e$ | 0.5329 | 2.3410 | 3.3045 | 3.6352 | 4.8726 | 4.8488 | 1.9758 | 0.2388 |
| Feigenbaum constant $\delta$ | 1.2318 | 1.8158 | 2.4026 | 1.7986 | 2.5540 | 1.8068 | 0.5228 | -0.2181 |
| Random | -0.8879 | 0.8467 | 1.8797 | 3.4841 | 3.0081 | 2.8593 | 1.1905 | -1.0196 |

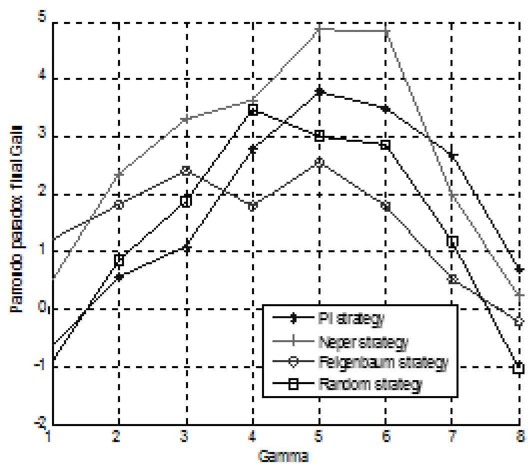


**Fig. 5.**The effect of $\gamma$ on the final gain of Parrondo's paradox for all switching strategies.

Fig.6 shows how gain changes across iterations in various strategies. It is clear that for $\gamma = 5$, the e strategy is significant different from random and all transcendental numbers strategies.

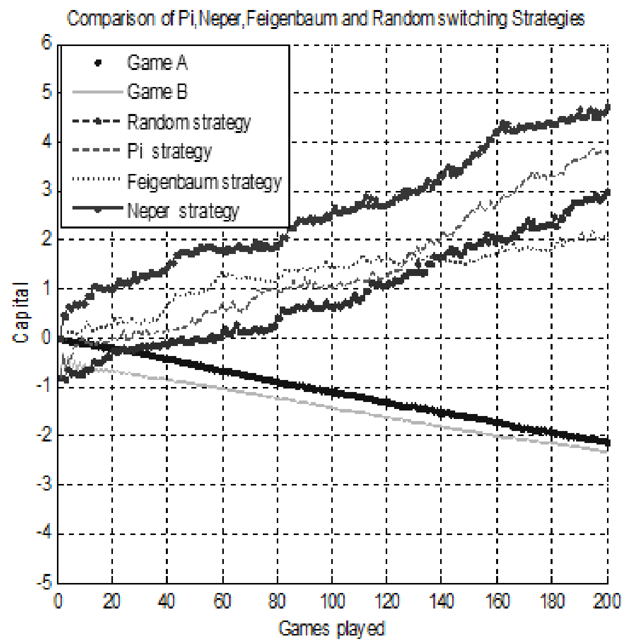


**Fig. 6.** Comparison of random and transcendental numbers $\pi$, $e$, Feigenbaum constant $\delta$ switching strategies in Parrondo's paradox game. Each point was obtained as the average over 50,000 trials.

**Table3.** Mean of gain variation after 200 iterations

| Strategy | $\gamma = 1$ | $\gamma = 2$ | $\gamma = 3$ | $\gamma = 4$ | $\gamma = 5$ | $\gamma = 6$ | $\gamma = 7$ | $\gamma = 8$ |
|---|---|---|---|---|---|---|---|---|
| $\pi$ | -0.9727 | -0.2533 | 0.0406 | 0.7241 | 1.3772 | 1.2328 | 1.0626 | 0.0390 |
| $e$ | 0.3987 | 1.2397 | 1.8948 | 2.3451 | 2.7378 | 2.8947 | 1.0951 | -0.2574 |
| Feigenbaum constant $\delta$ | 0.8523 | 0.9679 | 1.1774 | 0.8352 | 1.3014 | 0.9765 | 0.1674 | -0.2544 |
| Random | -0.2767 | -0.4988 | 0.0319 | 0.2745 | 0.7674 | 0.9080 | 1.0159 | -0.1459 |

As shown in table3, the mean of gain variations with $e$ switching strategy for $\gamma = 5$ is higher than the other switching strategies after 200 iterations.

## 3. CONCLUSIONS

In this paper a new application of transcendental irrational numbers $\boldsymbol{\pi}$, $e$ and feigenbaum constant $\boldsymbol{\delta}$ was investigated in the switching strategy of Parrondo's paradox game. In order to compare our new switching strategy to the Random strategy, time series of decimal numbers in three transcendental irrational numbers $\boldsymbol{\pi}$, $e$ and Feigenbaum constant $\boldsymbol{\delta}$ were used as a switching strategy in Parrondo's paradox game. Simulation results showed that except for random switching strategy, $\gamma = 5$ causes the maximum final gain in all strategies. Therefore it could be said that the digit 5 has a significant role in the three transcendental irrational numbers $\boldsymbol{\pi}$, $e$ and feigenbaum constant $\boldsymbol{\delta}$. Also by selecting the e switching strategy, Parrondo's paradox has a highest final gain in $\gamma = 5$ compared to another three strategies.

One of the applications of this new switching strategy which is based on irrational numbers is in studying social and biological models inspired by parrondo's games and in designation of some evolutionary algorithms. [11](e.g. Genetic algorithm or Cellular Automata ).

## 4. REFERENCES


[1] D.Lai, Marius-F. Danca, **Fractal and statistical analysis on digits of irrational numbers**, Chaos, Solitons and Fractals 36 ,2008, pp.246–252.

[2] F.Sezgina and T. Metin Sezgin, **On the statistical analysis of Feigenbaum constants** , Journal of the Franklin Institute, November 2006, pp.756-758.

[3] G. P. Harmer and D. Abbott, **Parrondo's paradox: losing strategies cooperate to win** , Nature 402,1999, pp.864.

[4] Harmer GP, Abbott D. **Parrondo_s Paradox**. Statistical Science, Vol.14, 1999, pp. 206–213.

[5] J. Daintith, Oxford Dictionary of Chemistry, oxford university press, 2008.

[6] J. Luo, **The Entropic Power of Energy and Natural Resources**, Massachusetts Institute of Technology, January, 2007.

[7] J. M. R. Parrondo, G. P. Harmer, and D. Abbott, **New paradoxical games based on Brownian ratchets**, Physical Review Letters 85(24),2000, pp. 5226–5229.

[8] K.J. Falconer, **Fractal Geometry. Mathematical Foundations and Applications**, Wiley, NewYork, 1990.

[9] M. F. Barnsley, **Fractals everywhere**, Boston: Academic Press;1988.

[10] Nathaniel F. G. Martin, James W. England, **Mathematical theory of entropy**. Encyclopedia of Mathematics and its Applications, vol. 12, Addison-Wesley Publishing Company, Reading, Mass., 1981.

[11] P. Arena, S. Fazzino, L. Fortuna, and P. Maniscalco, **Game theory and non-linear dynamics: the Parrondo paradox case study** , Chaos, Solitons and Fractals 17, 2003, pp.545–555.

[12] R. Hilborn, **Chaos and nonlinear dynamics**, Oxford University Press,2000.

[13] R. Toral, **Cooperative Parrondo's games**, Fluctuation and Noise Letters 1(1),2001, pp. L7–L12.

[14] T.W. Tang,a A. Allisona and D. Abbotta, **Parrondo's games with chaotic switching** , Noise in Complex Systems and Stochastic Dynamics II, 2004, pp.520-530.

[15]http://pi.lacim.uqam.ca/piDATA/feigenbaum.txt, http://infomotions.com/etexts/gutenberg/dirs/etext96/miscc10.txt and http://numbers.computation.free.fr/Constants/constants.html