

## Subtraction – the improvement of communication through critical aspects

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### ABSTRACT

The research study, this article is based on, aims to develop principles to enable the transfer of research knowledge to teaching. In this article are presented the design and implementations of a research project intended to support teachers in understanding their practice and improve it. Furthermore, the central idea was to improve the communication in mathematics between teachers and teachers-students. Issues that arose from practice were framed in terms of learning. In addition, the variation theory formed the background to our work. The students' tests, examination of students' mathematical work, the teachers' lessons plan and reports of the lessons' instructions are the data base for this article. The analysis indicated that teachers were not able to describe the critical aspects in students' learning in the beginning of the project. By giving teachers the training that allows them to become reflective teachers, they also get the possibility, as professional decision makers, to develop the ability to identify the critical aspects in students' learning and consider how opportunities for learning can be enhanced. Furthermore, if the teachers base their instructions on the identified critical aspects and open up dimensions of variation in these aspects, the students' learning seems to be facilitated. The findings suggest that developing an understanding of the students' critical aspects can be a productive basis in helping teachers to make a fundamental change in their instructions and to improve the mathematical communication in the classroom.

**Keywords:** Mathematics, potential aspects, critical aspects, variation theory, effective communication, reflective teacher

### 1. INTRODUCTION

Integrating new knowledge created through research with teaching has become an important area that needs quick attention with the growing emphasis on student learning activities. There are few research studies that report the changes or transformation in teachers' way to teach mathematics. In recent years, a number of international studies [e.g. 2, 9] have shown that the

process to integrate research with educational practice is slow. Several attempts to enable teachers to change their instruction have often been unsuccessful or are minor and inconsequential [e.g. 1]. Great deals of previous research [e.g. 2] speculate that fundamental change in teaching practice may be initiated by changes in teachers' knowledge. The main question is what kinds of knowledge would be changed.

Pang and Marton (2003) introduced the idea of the learning study. Learning study brings together a group of teachers who agree to investigate a lesson very detailed. They first plan the lesson, observe it being taught in the classroom, discuss and modify the lesson when necessary. A learning study is theoretically grounded. Lo Mun Ling's (2009) paper outlines three applications of variation theory to learning in the context of learning studies: students understanding of the object of learning (V1), teachers' understanding of the same object (V2) and variation as a pedagogical tool (V3). Learning studies are presented as a harmonious process in which teachers manage, on the basis of their observations, to reach a consensual decision about how the lesson needs to be modified for the next cycle.

Our experience (discussions with teachers and principals in the beginning of the project) is that the teaching and learning process (in a learning studies cycle) is difficult to sustain in a longer term. We found a tendency for the process to return into a traditional form of the daily accountability pressures teachers' face in their work. One major theme in our project is to give teachers the possibility to use the scientific knowledge and research in their practice thus giving them the training that allows them to become reflective teachers and create an effective communication in the classroom. To make easy this process for the teacher we have designed a more flexible procedure for a learning study which better corresponds to the organisational conditions in Swedish school and can be applied in a longer term.

### 2. THEORETICAL ASSUMPTIONS

On the basis of the article's aims, we will mainly focus on one major theoretical development concerned, namely the variation theory [4, 5], which relates the students' comprehension of a

specific content to the experience of the pedagogical situation in which it is met with. Runesson (2006) specifies that variation theory “is not a theory of the mechanisms of learning but a theory of the relation between the object of learning and the learner” (p.406). The object of learning is broadly regarded as “the complex of different ways of experiencing the phenomenon to be learned about” [5, p.162]. The objects of learning are the final points toward learning activities are directed and how learners understand them. Variation theory focuses on the way in which a phenomenon is made visible in a teaching context. The main idea is that in order to discern a difference, we must have experienced a variation from our previous experience. Marton et al. (2004) have defined four patterns of variations which can facilitate students’ discernment of critical aspects of the object of learning: (1) Contrast means that “in order to experience something, a person must experience something else to compare it with” (p.16); (2) Generalisation is to experience varying appearance of an object; (3) Separation of an aspect from other aspects is required; (4) Fusion is where several critical aspects need to be considered together.

To learn means to experience variations while to experience means to be aware of discerning certain aspects in a given context and relate them to this context. Moreover, only experiencing variation, which is a decisive condition for learning, can evoke discernment of these aspects. However, not all the aspects are significant for learning. A critical aspect of the object of learning contributes to a particular meaning in the learner’s awareness. Only variation in the critical aspects is an essential condition for learning [e.g. 6]. To help students learn such topics teachers must be able to understand why students may experience difficulties in discerning their critical features or aspects. In the teaching context the teacher develops the teaching material with a perception of the content, that is, an “intended object of learning”. Marton et al. (2004) argue that the object of learning is defined by “its critical features, that is, the features that must be discerned in order to constitute the meaning aimed for” (p.22). A critical feature is a way of “distinguishing one way of thinking from another” (p.24). The teacher can use appropriate variations within the identified space of learning to enact the object of learning [5]. What teachers/students learn constitutes the lived object of learning.

The concept of critical aspects develops in this article by introducing the division of the critical aspects into *potential and real critical aspects*. Potential critical aspects are what teachers believe to be critical aspects of students’ learning, while real critical aspects are what student’s exhibit as critical aspects in their learning. Those parts of

critical aspects generate the relationship between the intended and the lived objects of learning.

We find it useful to distinguish between two broad categories of aspects because these facilitate to define what means by reflective teachers. *Reflective teachers* seek to probe beneath the appearance of a commonsense reading of experience potential critical aspects in their teaching. They investigate the real critical aspects and become aware to improve students learning.

The variation in the ways teachers/students experience the object of learning they meet in their area of activity can be analysed and described in terms of a small number of different categories determined qualitatively. Among these categories, teachers can identify the features that are important for current understanding, possibly not as comprehensive as the teacher’s own understanding but adequately powerful for the current concerns. By being able to open up dimensions of variation in the context of the categories, the teacher is also able to identify ways of continuing to a more sophisticated understanding and to ensure that current understanding also contains the germs of even more powerful ways of understanding for future needs.

The concept of variation is not “an eclectic approach” or “diverse” organization, for example sometimes to work practically or thematically, you either let students work in groups or individually. It means different ways to vary the content - the different variation patterns: contrast, generalisation, separation and fusion.

We defined the *effective communication* as a process by which the teacher assigns and conveys meaning in an attempt to create shared understanding, i.e. the process of meaningful interaction among the intended, enacted, and lived objects of learning. In analysing the effectiveness of communication, endeavours to assess how the intended, enacted, and lived objects of learning compare, that can be discerned. The intended object of learning can be compared to the categories of description relevant to the phenomenon in order to determine both the level of awareness being focused on and the appropriateness of the intended object of learning. This could be considered an initial assessment of whether the teaching is intended to target the appropriate level of learning. The intended and enacted objects of learning can be compared to determine whether what is being taught matches to what was intended to be taught. The students’ lived object of learning can be compared against categories of description as a means of assessing the level of learning achieved or against the enacted level of learning to determine whether the enacted object of learning is being transferred to the lived object of learning as expected.

### 3. METHOD

From 2007 to 2010, we studied the teaching and learning of the number sense, counting, pre-algebra, equations, formulae, functions and transformation of algebraic expressions from pre- to upper secondary school. Over a 3-year period the teachers participated in a development project (Critical aspects as basis for development of mathematics teaching and students' learning) that focused on helping teachers to identify the critical aspects in students learning and decided how to use that knowledge to make dimensions of variation in these aspects to improve the students' learning. This 3-year study investigated 22 teachers' instructions as they learned. The research was working in collaboration with the class teacher. The teacher agreed to participate in the study reported in this article with the understanding that feedback was given to the school about the student's understandings of mathematical concepts thus these information's could be used to plan future teaching approaches.

An overview of the qualitative approach to data source, collection and analysis, the theoretical perspectives, and reliability can be seen in Figure 1.

Data sources	Data collection techniques	Data analysis	Reliability
teacher	teachers description of potential critical aspects	construct potential critical aspects profiles	Scoring of potential critical aspects
	interviews	construct potential critical aspects profiles	
	teachers' lessons plan	identifying the object of learning	verification of the object of learning with the plan of the lessons
	teachers' report	identifying patterns of variation	verification of the dimensions of variation with lesson reports
students	students' tests	identifying real critical aspects	verification of real critical aspects
	interviews	identifying real critical aspects	

Figure 1. Overview of the study approach

The analyses of teachers' description of potential critical aspects have been divided into distinct categories based on the aspects they focus on. The scoring of these aspects was carried out by comparing the number of sentences in each category with the total number of sentences that teachers took up. If there were confusions about certain expressions, group interviews with the teachers, they were carried out. The collection and analysis of data consist of three phases. The relations between those are shown in Figure 2.

The project began by explaining various concepts used in the variation theory to the teachers and putting those concepts into practice. Then the teachers worked to identify potential critical aspects in students' learning. Subsequently tests and interviews were conducted with students to identify the real critical aspects of their learning. The material was analyzed and differences between potential and actual critical aspects were identified (Phase 1). In the second phase, each teacher had to plan and implement 6 lessons focusing on the real critical aspects identified in phase 1. After each

lesson, a detailed report was made after the following template: (I) General information: school, class/group, teacher, moment, object of learning, type of lesson; (II) General purpose; (III) Specific purpose: content, emotional view, psychomotor view; (IV) Prerequisites: technical aids, materials; (V) Teaching method; (VI) Activities with students; (VII) Lesson implementation according to the following: didactic moments, teacher's activity, student's activity. The implementation of lessons ended with assessing students' learning using tests or interviews. The analysis of the collected materials focused on the identification of the critical aspects of the teachers, opening up the dimensions of variation and classifying them.

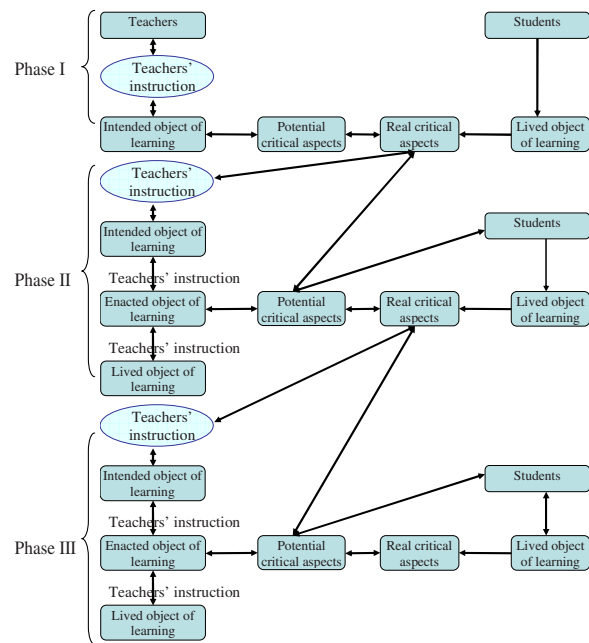


Figure 2. Overview of phases

There were 6 lessons based on the results of Phase 2. Teachers planned the implementation of the lessons and after each lesson they made a detailed report using the same template as in Phase 2. Teachers create the report to communicate their instructional activities regarding specific subject-matter and describing the variations opened in the identified critical aspects. Almost all reports developed by teachers contain information of students learning. The analysis of teachers' reports was conducted in the same manner as in Phase 2. This article will present only the analysis of phase 1 and 2 when subtraction is presented in the classroom. The material gathered in phase 3 is in the process of being analysed.

Parents of all students were informed about the research and their right to withdraw their child from participation in the research at any time. No parent chose to withdraw a child from this research. Students were also informed in simple terms about the reasons for the researcher's presence in the

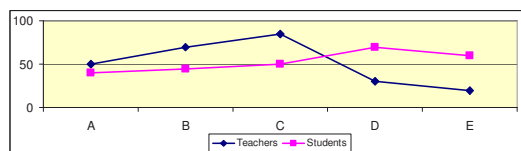
classroom and for interviews, and were given the choice of whether they wanted to participate or not.

#### 4. RESULTS

On way to demonstrate the interaction between the potential/critical aspects, reflective teacher and effective communication is to point out what happens with a learning object (subtraction) for 4 primary grade teachers (grade 1, 2, 3 and 4).

##### Phase 1

By analysing the teachers' reports and students' tests, 5 distinct categories could be identified: the whole (A), the parts that form the whole (B), the relation between the parts (C), the transformation between the parts (D) and the relation parts-whole (E). The teachers' written reports were completed by group interviews when unclear expressions were encountered. The results are presented in Figure 3.



**Figure 3.** The potential and real critical aspects

In Figure 3 we can see that there is a difference between the potential and the real critical aspects. To a large extent teachers believe that students do not understand a task that contains subtraction as a whole (A), the numbers constituting parts (B) and the parts related to each other by subtraction (C). However, they do not believe that students need a better understanding of how to relate the parts to each other in a different way, i.e. the transformation between the parts (D) and the relation between parts and the whole (E). Despite this, the result in Figure 2 shows that about 50% of students do not understand the first three relationships and more than 50% of the students do not understand the last two relationships.

The results of students' understanding of subtraction are worrying. The following extract from interviews shows that first year students do not discern the relation between parts and the way in which the parts constitute the whole:

Teacher: Look here now then. I have 1, 2, 3, 4, 5, 6 crayons in my hand. Now I put 2 crayons in the box. How many do I have left in my hand? (The teacher holds his hand behind his back.)

Student: 4

Teacher: How do we say this in math language?

Student:  $2-4 = 6$

Teacher: Oh, do you know what are you saying now? If we have two crayons, we can hide the fourth in the box. Is it possible?

Student: No

Teacher: No that was not what happened.

Teacher: How many crayons did I have in my hand?

Student: 6

Teacher: How do we write?

Student:  $4-2 = 6$

This phenomenon persists in the year 2, 3 and 4, being the basis for the children understanding that the larger number comes first when writing subtraction. Students who are finding subtraction difficult often make that particular error. They aren't well grounded in the concept that the larger number is written first – a dilemma which can create problems turning into a mystery for them. That can be seen in the following example:

Malin saves money to buy a bike that costs 525 Swedish Kronor. She has 378 Swedish Kronor. How many more does she need before she can buy the bike?

5	0	9	-	3	0	=	2	0	0
7	8	-	2	0	=	5	0		
8	-	5	=	3	=	2	5	3	4

**Figure 4.** Markus (4th year student)

Markus shows that he understands the text and can see how 378 are related to 525. In addition, Markus discerns the parts constituting the whole, but he can not relate these parts to each other in a different way and to the whole. The analyse shows that 55% of the students have the same critical aspects.

It is necessary to specify that the teachers implemented the teaching of mathematics without using the new concept of variation theory as theoretical perspective. In the beginning of the project the focus in the teachers' training was only the concept of critical aspects.

##### Phase 2

Based on the identified real critical aspects in students learning and the difference between potential and critical aspects after the first phase, the key concept of the theory of variation was taken up again. In our discussions, we found that it is always beneficial to give students experience in linking subtraction facts with addition facts. This is because a student's memory of both addition and subtraction facts helps the student to understand that each subtraction fact is related to an addition fact. To do this, students need to understand the differences and the connection between addition and subtraction. This means that students need to understand how the parts relate to each other and that the relationship can take many forms. For example, if a child knows that  $9 + 8 = 17$  and  $8 + 9 = 17$ , he or she easily learns that  $17 - 9 = 8$  and  $17 - 8 = 9$ . We have found the following two ways for providing experience in teaching the relation between addition and subtraction facts is very



useful and interesting. Besides these, we have identified that students perceive that the larger number comes first when writing subtraction.

On this basis, the teachers implemented 6 lessons. The analysis of teachers' reports shows both the critical aspects that they focused on and the dimensions of variation that they opened up in these aspects. The teachers' focus was for the students to understand: (1) the meaning of the term addition; (2) that a number or an amount can be reduced; (3) understand the difference between addition and subtraction; (4) recognizing the minus sign as a symbol and understand its meaning; (5) be able to decide whether to use addition or subtraction to solve the task; (6) be able to express the addition or subtraction so that the terms are written in the right order; (7) the direction in which they read the subtraction; (8) be able to see a connection between the subtraction and addition; (9) the principle of "tens of themselves and units alone" does not always work; (10) the need to have "an eye on" units in a subtraction; (11) to know that the commutative law is not applicable to subtraction (be able to discern the difference between e.g. 5-3 and 3-5).

In these aspects the teachers opened up dimensions of variation by contrast (3, 5, 6, 11), separation (1, 2, 4, 10), fusion (8) and generalisation (7, 9). Besides these, the analysis of teachers' reports shows that they open up a new dimension of variation (subordinate to generalisation) that will be called *similarity*. The dimension of variations named similarity is defined as the property of two or more expressions to adapt the same meaning.

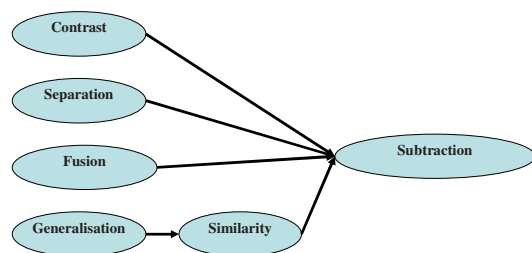


Figure 5. Dimensions of variation

In the teachers' questions it was identified that they vary the following expressions: left (How much money is left? How much money do you still have? How much cash did you get back? How much change did you get?), missing (How much is missing?), lost, got out, gave away, escaped, sell, left over, less, difference and the opposite. This variation makes it possible for students to develop an understanding from explicit expressions used to mark the operation they have to apply as well as to understand subtraction as the difference. All these opened dimensions of variation are orientated to the meaning of subtraction and how this is expressed.

The analysis of students' tests and interviews show that students could understand the differences and the connection between addition and subtraction. For example, the same test was conducted with students in year 2 (involved in the project) and students in year 3 and 4 (which did not participate in the project). The results show that students in year 2 perform better than students in the year 3 and 4. The solution rate indicates that 60% of the students in the year 2, 39% of the students in the year 3 and 37% of the students in the year 4, could solve the exercises that were in the test. Moreover, 95% of the students in the year 4 could solve the task presented in Figure 4.

In the analysis of teachers' reports and students tests/interviews, the same categories as in phase 1 were identified. The relation between the aspects focused on in the classroom and the critical aspects in students learning are shown in Figure 6.

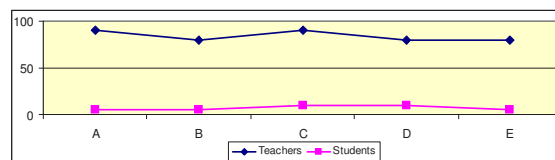


Figure 6. The potential and real critical aspects

The results show that teachers did not only focus on the identified critical aspects in the project's first phase, but they focused on and opened up dimensions of variation in all categories. This led to a reduction in the critical aspects of students' learning.

## 5. CONCLUSIONS

Four important discoveries were identified in this study. First, the link between research and teaching is not automatic. The research enquiry helped teachers to see their workplaces and the possibilities for the action within themselves having new theoretical perspectives. Illumination can occur when teachers simply use research to assist their interpretations of students' learning. It can also happen when teachers themselves undertake research and in doing so they use the lenses offered by published research, thus they can systematically examine and develop their own practices and the context in which they are working. In both cases teachers are theorising their practices and exploring the potential available for their deliberate actions. Teachers need the lenses that research can give them for both recognition and analysis of the object of learning in practice. We know that the experts in all activities differ from the novices in the way which they rapidly scan fields, identify what is significant and respond to quite complex interpretations at a level which is sometimes described as intuitive. For teachers to get the opportunity to become experts, they need to focus

in a way which they use the research when scanning, interpreting and responding to learners and learning content. The most important change in this project is to design and implement the research findings to improve the teaching while keeping in mind that learning must be improved as well. A good teaching-learning connection provides active experiences that give students many opportunities to experience new material, and work toward mastery. The critical aspects and variation in those aspects, therefore, are a primary factor in encouraging teacher to improve teaching and students' learning. In order to understand what variations to use in the classroom to improve students learning, it is necessary to understand the varying ways of experiencing the object of learning. Fennema and Franke (1992) named this "changes in teachers' knowledge". In this way it is possible to integrate research with educational practice, in contrast with other researchers' discovery [e.g. 1, 9].

Secondly, the objects of learning intended by the teachers differ from students' lived object of learning (see Figure 3). The teachers' intentions are to enact an object of learning on what students already know instead of focusing on what they do not know. This implies an inefficient communication in the classroom because the students do not encounter an object of learning in the way that is necessary to develop their learning. This is due to differences between potential and real critical aspects. We found that an object of learning can be analysed in a general way in five categories: the whole (A), the parts (B), the relations between the parts (C), the transformations between the parts (D) and the relation between the parts and the whole (E). The most critical aspects in students' learning appear in categories C, D and E. By reflecting to these general categories, the teachers constituted a complete learning object in the sense that they were able to take up almost all critical aspects in the students' learning. This resulted in an essential improvement of student learning. Subtraction is often the point where students become fearful of mathematic and lose their confidence in their ability to master mathematical concepts. It doesn't have to be that way. We also found that the teachers open dimensions of variations in all categories to overcome the critical aspects in categories C, D and E. A new pattern of variation was identified. The dimension of variations named *similarity* is defined in the following way: the property of two or more expressions to adapt the same meaning.

Thirdly, the teacher may enact an object of learning in a lesson that does not express the students need. This means that communication in the classroom is not effective. Teachers' analysis of the relationship between the potential and the real critical aspects enables them to develop the way of

experience of the mathematical object of learning and find one correlation between the intended, enacted and lived object of learning. These correlations seem to lead to effective communication in the classroom.

Fourthly, to implement a lesson plan in the relation to the report of the lesson with focus on creating dimensions of variation in the critical aspects of the content seems to be a powerful tool to the teachers' reflective process.

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