Pentahedral Honeycomb with Skew Hexagonal Faces

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ABSTRACT

This paper proposes a generally new type of macro and megaspace filling honeycombs having quasiregular pentahedral cells with skew hexagonal faces. An existence of spatial cells named pentahedra is demonstrated by topological transformations of hexagonal prismatic honeycomb and is based on a recently discovered *Phi* relationship within a regular hexagonal tessellation. A geometric symmetry of this honeycomb is studied, filling the space with no gaps or overlaps. Finally it is pointed out that abstract or skeletal analogues of pentahedral honeycomb have effective practical uses by synthesis of artificial man-made macromedium, especially the like of orbital large scale structural systems.

Keywords: Topological Transformations, Skew Faces, Pentahedral Honeycomb

1. INTRODUCTION

Two earlier made discoveries lie in the ground of necessity to carry out these investigations within spatial macro structural geometry.

The first lies within a structural mechanics. In 1993 it was demonstrated that stiffness components of the topological invariants of a spatial bar system are not identical by identically used volume of the material [2]. The numerical value of this characteristic is the most highest of cell like a lattice of diamond, exceeded for example 5.9 times a topological stiffness component of a traditionally used triangular lattice.

It had been discovered in 2007 that vertices of regular hexagonal tessellation are Phi centres with very low variance that corresponds to the 11-th series of the Fibonacci convergence sequence on Phi [3, 4]. It means that an existence of such a rational geometric ratio could secure the highest specific mechanical stiffness exactly of the hexagonal structure. Thereby here is a geometrical problem to construct a honeycomb like a spatial analogue of planar regular hexagonal tiling.

2. TOPOLOGICAL TRANSFORMATIONS OF A HEXAGONAL PRISMATIC HONEYCOMB

We start with building up an infinite horizontal layer of regular hexagonal prisms (Fig. 1). Let a be the length of edges of their base hexagons, and let y be the height of each prism, i.e., the distance between the horizontal base planes (it will be determined below).

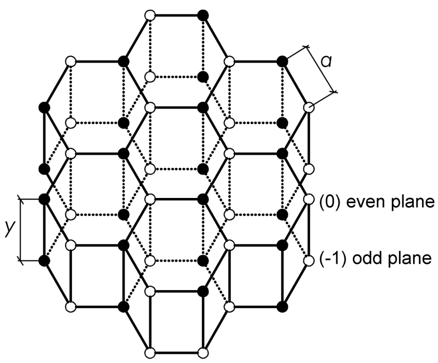


Figure 1: A layer of hexagonal prisms:
(0) and (-1) – even and odd horizontal planes of vertices, respectively; a – length of edges of hexagons;
y – thickness of a layer.

We further make up a hexagonal prismatic honeycomb by arranging layers one above the other so that bases of prisms in neighbouring layers coincide. The horizontal planes between layers are labelled by consecutive integers: …, -4, -3, -2, -1, 0, 1, 2, 3, 4… We also colour vertices of the grid in black and white so as ends of every horizontal edge are coloured differently, while ends of every vertical edge have the same colour. This colouring is used to describe a certain deformation within the layer: (a) we move all black points lying in the planes labelled by even integers (including 0) vertically up by an amount x, and all the white points down by the same amount x (x will be determined later on); (b) we act the other way (black vertices down, white ones up) in the odd planes (Fig. 2).

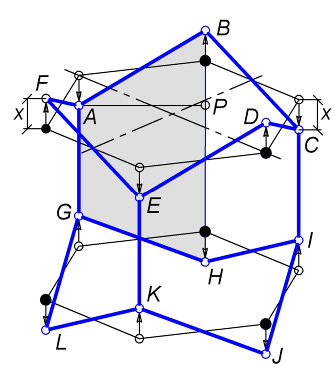


Figure 2: Topologic transformations of hexagonal prism:
x – amount of the vertical displacement of vertices.

During the topological transformation vertical edges of former hexagonal prismatic honeycomb are divided in two groups such that one group consists of those of increased length ($y+2x$) and the other group – of reduced length (y-$2x$). It means that instead of 6 vertical rectangles of each former hexagonal prism we have obtained 3 regular once convex hexagonal faces, but instead of 2 horizontal hexagons, 2 regular twice convex faces. In this way, we obtain a peculiar polyhedron of new kind with five skew hexagonal faces which will be called pentahedron.

We are now going to determine the corresponding values of the parameters x and y. Let us observe one vertex O within obtained pentahedral honeycomb (Fig. 3) connected with vertices J, N, C and H. In this figure, the point I was moved up, and the points J, N, and H were moved down from their common plane, while the point C was moved down from the nearest upper plane. The point I has now only four adjacent vertices (in contrast to five in the former hexagonal prismatic honeycomb). As we want the four edges incident to I to be of equal length, and the six angles between those edges of the same size, the pyramid $JNHC$ has to be regular and the point I has to be its focus.

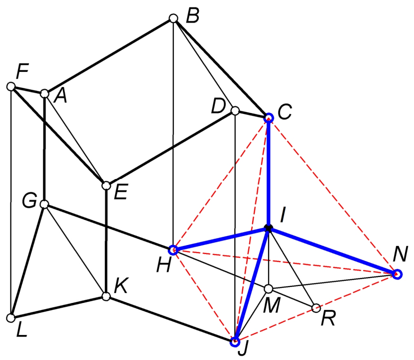


Figure 3: The pentahedral honeycomb:
regular tetrapod circumscribed by
triangular pyramid.

Let us draw a regular triangular pyramid $JNHC$ with focus I, denote the centre of its base triangle by M, and the midpoint of the edge JN by R (Fig. 3). Recall that a was the initial length of edges of hexagons, x was the amount of the vertical shift of vertices, and y was the thickness of a layer (i.e., the length of vertical edges). Therefore $IM = NM = HM = a$, $IM = 2x$, and $MC = y$.

Since the pyramid is regular, the edges of pentahedron incident to I are of equal length, which we denote by z:

$$JI = NI = HI = CI = z\ .$$

The angles between these lines also are equal; let α stand for their size. Then

$$\angle JIN = \angle NIH = \angle HIJ = \angle JIC = \angle NIC = \angle HIC = a\ .$$

Next we will determine the values of the above-defined parameters x and y so that the pyramid $JNHC$ could be regular, indeed. Using Eg. (1) - (8) and performing the calculations, we finally will also obtain the value of z with respect to y and x as well as the size of the angle α:

$$JN = NH = JH = JC = NC = HC = \sqrt{3}a \qquad (1)$$

$$y = MC = \sqrt{NC^2 - MN^2} = \sqrt{3a^2 - a^2} = \sqrt{2}\,a \qquad (2)$$

$$IM = \frac{1}{4}MC = \frac{1}{4}\sqrt{2}a \qquad (3)$$

The focus of a regular pyramid divides its height in the ratio 1:3. Then

$$x = \frac{1}{2}IM = \frac{1}{8}\sqrt{2}a \qquad (4)$$

$$IR = \sqrt{IM^2 + MR^2} = \sqrt{\frac{a^2}{8} + \left(\frac{MN}{2}\right)^2} = \sqrt{\frac{a^2}{8} + \frac{a^2}{4}} = \sqrt{\frac{3}{8}}a \qquad (5)$$

$$NR = \frac{1}{2}JN = \frac{\sqrt{3}}{2}a \qquad (6)$$

$$a = 2\angle RIN = 2\arctan\frac{NR}{IR} = 2\arctan\left(\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{8}}{\sqrt{3}}\right) = \\ = 2\arctan\sqrt{2} \approx 109{,}47° \qquad (7)$$

$$z = JI = \sqrt{JM^2 + IM^2} = \sqrt{a^2 + \frac{a^2}{8}} = \frac{3}{\sqrt{8}}a = \frac{3}{4}\sqrt{2}\,a \qquad (8)$$

Summing up: pentahedral honeycomb is a structure determined by a single parameter z. In order to construct this structure, a hexagonal prismatic honeycomb has to be constructed first. In this latter one the length of an edge of each hexagon is $a = \frac{4}{3\sqrt{2}}z = \frac{2}{3}\sqrt{2}z$, the height of each prism (i.e., the length of its lateral edges) is $y = \sqrt{2a} = \sqrt{2}\frac{2}{3}\sqrt{2}z$.

Then vertices of the prisms have to be shifted by $x = \frac{\sqrt{2}}{8}a = \frac{\sqrt{2}}{8}\cdot\frac{2}{3}\sqrt{2}z = \frac{1}{6}z$. In this way, the faces of prismatic hexagonal honeycomb are transformed into skew hexagonal faces of the pentahedral honeycomb. All the angles between the adjacent edges in the pentahedral honeycomb will be $a = 2\arctan\sqrt{2} \approx 109.47°$.

Let us check one more geometrical relation within a pentahedral cell having skew hexagonal faces. In Fig. 2 depicted is one deformed lateral face $ABHG$ of an initial hexagonal prism. It is an equilateral trapezoid; the sides AG, GH, and AB are edges of a cell in the newly constructed pentahedral honeycomb, $\angle AGH = \angle GAB = a = 2\arctan\sqrt{2} \approx 109.47°$ and AP is the height of the trapezoid. We should to check that $BP = 2x = \frac{1}{3}z$.

Indeed,

$$BP = AB\cdot\sin\angle PAB = AB\cdot\sin(\angle GAB - \frac{p}{2}) = \\ = z\cdot\sin(2\arctan\sqrt{2} - \frac{p}{2}) \approx 0{,}333333\cdot z \approx \frac{1}{3}z.$$

3. SYMMETRY OF THE PENTAHEDRAL HONEYCOMB

Symmetry, being the most inherent property of pentahedrons and spatial pentahedral tessellations, has been studied both on the level of a separate cell and of a honeycomb as a whole by means of symmetry groups (Fig. 4).

The pentahedron has 12 vertices, 15 edges, and 5 faces – the skew hexagons $ABCDEF$, $GHIJKL$, $ABCIHG$, $EDCIJK$, and $AFEKLG$. In a pentahedral honeycomb, every vertex is incident to 4 equal edges, and the angle between any two of them is $a = 2 \cdot \arctan\left(\sqrt{2}\right) \approx 109{,}47°$.

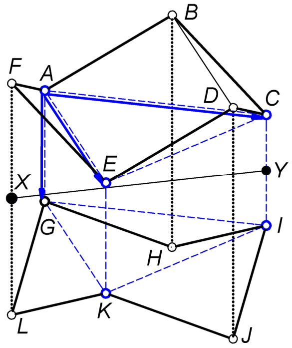


Figure 4: On a symmetry of pentahedral cell: .... bend lines of skew hexagonal faces with midpoints *X* and *Y*; triangular prism, vectors of the symmetry.

3.1 Polyhedral Symmetry Group

The symmetry group of a pentahedron, i.e. the group of orthogonal transformations of a space, that map the figure onto itself, is isomorphic to the symmetry group D_3 of a regular triangular prism. This follows from the fact that every orthogonal transformation which maps a pentahedron onto itself is completely determined by an orthogonal transformation which maps onto itself the regular prism *ACEGIK*. The following transformations are the generators of the symmetry group of a pentahedron:

a) Rotation through 120° on the central vertical axis of the pentahedron,
b) Symmetry about the central horizontal plane,
c) Symmetry about the line *XY*.

3.2 Symmetry of the Honeycomb

There are three independent vectors the translation along which maps a honeycomb onto itself. These translations are the base elements of the translation group of the honeycomb which is isomorphic to the free Abelian group $Z^3 = Z \oplus Z \oplus Z$. More specifically, the base consists of

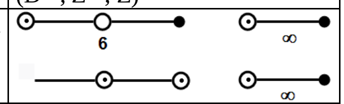
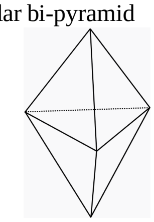
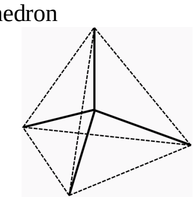
a) The vertical translation along the vector $\overrightarrow{AG} + \overrightarrow{FL} = \overrightarrow{AG} + \frac{5}{3}\overrightarrow{AG} = \frac{8}{3}\overrightarrow{AG}$,
b) The translation in the horizontal plane along the vector $\overrightarrow{AC}$,
c) The translation in the horizontal plane along the vector $\overrightarrow{AE}$.

The (full) symmetry group of the honeycomb is generated by

a) Generators of D_3,
b) Generators of Z^3, and
c) The rotation of the honeycomb through 60° on the vertical axis of a pentahedral cell.

An infinite pentahedral tessellation includes space tunnels structures with skew hexagonal faces. They have six-fold spiral symmetry, for polyhedrons packed along vertical axis repeat both after a rotation through 60° and a translation by a distance equal to the length of each edge. The main characteristics of symmetry properties of a pentahedral honeycomb are given in Table 1.

Table 1. The main characteristics of symmetry properties of a pentahedral honeycomb.

No	Characterization	Explanation
1.	Type	Convex uniform honeycomb
2.	Family	skew hexagon faces polyhedra
3.	Cell type	{6.4}
4.	Face type	{6}
5.	Schläfly symbol	{6.6}
6.	Coxeter group	(D^3, Z^3, Z)
7.	Coxeter – Dunkin diagram	
8.	Cells / edge	{4.3}4
9.	Faces / edge	63
10.	Cells / vertex	{4.3}
11.	Faces / vertex	66
12.	Edges vertex	6
13.	Dual	triangular bi-pyramid 
14.	Vertex figure	tetrahedron 
15.	Internal angle	109.47°
16.	Symmetry group	D3
17.	Other properties	isogonal and isotoxal polyhedra, n=6 fold helical symmetry of Z tunnel

4. ON A SKELETAL APPROACH TO THE PENTAHEDRAL HONEYCOMB

The long standing challenge of designing and constructing new crystalline solid state materials from molecular building blocks had been successfully started [6]. This success concerning a reticular synthesis (or chemistry) of robust materials with highly porous frameworks and with predetermined chemical properties had been achieved by investigation of abstract (skeletal) micro and nanostructural polyhedra.

This same conceptual approach could be employed for the creation of non-default macro and megastructural skeletal systems by predetermining such mechanical properties like minimum mass or maximum stiffness. Branko Grünbaum made a special study of abstract polyhedra, in which he developed an early idea. He defined a face as a cyclically ordered set of vertices, and allowed faces to be skew as well as planar [5]. Moreover in modern computer graphics any polyhedron gives rise to a graph or skeleton with corresponding vertices and edges.

It has been proved in a Chapter 2 that macrospatial pentahedral honeycomb has the same skeletal graph or skeleton as a nanospatial lonsdaleite or hexagonal diamond, which vertices are like junctions of tetrapod shape. It allows creating structural building constructions like hybrid bar systems. Thereby tetrapod shape junctions of bars or finite superelements of this system must be shells without moments of deflection like the most effective macro and megaconstruction from the point of view of used volume of material [1].

5. CONCLUSIONS

1. The pentahedral honeycomb has been obtained by topological transformations (stretching) of a regular prismatic hexagonal tessellation and is quasi-regular (vertex and edge transitive) with cells having three regular once convex and two regular twice convex faces.
2. Pentahedral honeycomb is a third type of discrete symmetry groups equipped with a topology or an infinite space group which combines elements of both point groups and lattice groups and also include such an extra transformation like screw axis.
3. Pentahedra are homeomorphic to hexagonal prisms uniform polyhedra consisting of regular skew hexagonal faces and congruent vertices. So they have exactly the same size and shape and are second space tiling polyhedra after cubic one, tilling space without holes and overlaps.
4. It could foresee a widely use of pentahedral lattices for a synthesis of minimum mass and maximum stiffest large scale structural especially orbital systems or builds on Earth natural satellites.

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REFERENCES

[1] E. Bervalds, *Hybrid Bar Constructions of Steerable Mirror Antennae.* **Papers of URSI Riga Meeting**, Vol. 1, September 4-6, 1990, 125-133.
[2] E. Bervalds, *Topological Transformations and Design of Structural Systems*. **Proc. of the World Congress on Optimal Design of Structural Systems**. Rio de Janeiro, Brazil, Vol. 1, August 2-6, 1993, pp. 153-160.
[3] E. Bervalds, *Existence of the Phi relationship within a regular hexagonal tessellation*. **Latvian J. of Physics and Techn. Sciences**, 2007, No. 2, pp.57-58.
[4] E. Bervalds, M. Dobelis, *Geometric Properties of a Regular Lattice caused by Phi Relationship*. **Proc. of the 13th International Conference on Geometry and Graphics**. Dresden, Germany, August 4-8, 2008, 8 p.
[5] B. Grünbaum, *Are Your Polyhedra the same as My Polyhedra.* **Discrete and Computational Geometry**. The Goodman – Pollack Festschrift, eds. Springer, New York, 2003, pp. 461-488.
[6] O.M. Yaghi, M. O'Keeffe, N.W. Ockwing, , H.K. Chae, M. Eddaoudi, J. Kim, *Reticular synthesis and the design of new materials*. **Nature**, (423), 12, 2003, pp. 705-713.