A Generalized Definition of Jacobian Matrix for Mechatronical Systems

Hermes GIBERTI, Simone CINQUEMANI
Mechanical Engineering Department, Politecnico di Milano, Campus Bovisa Sud, via La Masa 34, 20156, Milano, Italy

Giovanni LEGNANI
Dipartimento di Ingegneria Meccanica ed Industriale
Università degli Studi di Brescia, via Branze 38, 25129, Brescia, Italy

ABSTRACT
Manipulator kinetostatic performances are usually investigated considering only the geometrical structure of the robot, neglecting the effect of the drive system. In some circumstances this approach may lead to errors and mistakes. This may happen if the actuators are not identical to each other or when the employed transmission ratio are not identical and/or not constant.

The paper introduces the so called “Generalized Jacobian Matrix” obtained identifying an appropriate matrix, generally diagonal, defined in order to:
1. properly weigh the different contributions of speed and force of each actuator.
2. describe the possible non-homogeneous behaviour of the drive system that depends on the configuration achieved by the robot.

Theoretical analysis is supported by examples highlighting some of the most common mistakes done in the evaluation of a manipulator kinetostatic properties and how they can be avoided using the generalized jacobian matrix.

1. INTRODUCTION
The behaviour of a serial or parallel manipulator can be investigated through its kinetostatic performances [1] such as repeatability, stiffness, maximum force or velocity. They all depend on the kinematic structure of the system, on its configuration in the working space and on the kind of drive system used to operate the robot.

The manipulator may have singular configurations in which the performances in some directions are extremely poor while in others are extremely good. Conversely the manipulator may have configurations where the performances are identical in all directions. This behaviour can be described through the concept of isotropy [2], [3]. Naturally, the design of an isotropic machine is desirable because it assures homogeneous performances in all the directions in terms of accuracy, repeatability, stiffness, maximum force and velocity [4].

The kinetostatic properties of a manipulator, as a function of its position in the workspace, can be analyzed through the jacobian matrix \( J \) [5] or by the ellipsoids of manipulability, strictly related to the jacobian matrix itself [1].

Generally evaluation of isotropy, however, is carried out under the assumption that the behaviour of all the actuators is independent by the pose of the robot and it is the same for all the actuators. This assumption corresponds to exclude the effects of the drive system on isotropy, thus assuming that it depends only on the robot geometry and its reached position [6]. This practice inevitably leads to an incorrect formulation of the problem and to an inaccurate assessment of the isotropy of the system [2].

The paper deeply analyzes this problem introducing the definition of Generalized Jacobian Matrix \( J^\ast \): unlike the jacobian matrix, it allows to evaluate the real isotropy of a manipulator taking into account the effects of the drive system on the performances of the robot.

2. MANIPULATOR ISOTROPY
Robot performances are usually measured referring to jacobian matrix \( J \). The function:
\[
f(x, q) = 0
\]
shows the relationships between the joint space coordinates \( q \) and the workspace ones \( x \). Differentiating eq.(1) one gets:
\[
J_x \dot{x} = J_q \dot{q}
\]
where:
\[
J_x = \frac{\partial f}{\partial x}; \quad J_q = -\frac{\partial f}{\partial q}
\]

The jacobian matrix \( J \) can be expressed as:
\[
J = J_q^{-1} J_x
\]
linking velocities of joint space \( \dot{q} \) with the workspace ones \( \dot{x} \) as:
\[
\dot{q} = J \dot{x}
\]

Thanks to the so called kinetostatic duality [...] the transposed jacobian matrix represents the relationship between the forces and torques acting on the end-effector \( F_e \) and forces and torques exerted by actuators \( F_q \):
\[
F_e = J^T F_q
\]

Kinetostatic properties of a manipulator, as a function of its position in the workspace, can be analyzed through some indices related to the jacobian matrix [2,5]. Isotropy is one interesting property of a manipulator, since it defines the behaviour of the robot along each direction.

Remembering that the \( i \)th singular value \( \sigma_i(A) \) of a matrix \( A \) is defined as the square root of the eigenvalue \( \lambda_i \) of the corresponding matrix \( A^TA \):
\[ \sigma_i(A) = \sqrt{\lambda_i(A^T A)} \] (7)

where \( \lambda_i \geq 0 \), isotropy can be “measured” through index:

\[ I = \frac{\sigma_{\max}}{\sigma_{\min}} = \text{cond}(J) \] (8)

which is the condition number of the jacobian matrix.

When it is verified \( \text{cond}(J)=1 \), the minimum and the maximum eigenvalues coincide and the manipulator is defined as isotropic.

Condition on isotropy can also be expressed as [5]:

\[ J^T J = kI \] (9)

where \( k \) is a scalar and \( I \) is the identity matrix. That means isotropy can be achieved when jacobian matrix is proportional to an orthogonal matrix.

This definition, however, is carried out under the assumption that the behaviour of all the actuators is independent by the pose of the robot and it is the same for all the actuators.

This assumption corresponds to exclude the effects of the drive system on isotropy, thus assuming that it depends only on the robot geometry and its reached position. Moreover this classical definition does not consider that some of the gripper and joint coordinates describe rotations and other describes translations and so utilises different units (e.g. degrees and meters).

This practice inevitably leads to an incorrect formulation of the problem and to an inaccurate discussion of the properties of the system like isotropy.

To overcome these problems it is possible to introduce some “characteristic lengths” utilized to normalize the dimension of the manipulator; one length is used to correlate rotation of the TCP with its translation, while a second length is used to compare revolute and prismatic actuators [7]. However the choice of the value of these parameters is arbitrary and some criteria to select reasonable values should be developed.

The paper deeply analyzes the problem of comparing different actuators introducing the definition of Generalized Jacobian Matrix: unlike the jacobian matrix, it allows to evaluate the real isotropy of a manipulator, taking into account the effects of the drive system on the performances of the robot.

To better understand these concepts, a case study is presented: it is a 5R 2 dof parallel kinematic machine consisting in 4 links (5 considering the ground) connected by five revolutionary joints (R) two of which are located on ground and driven by motors (Fig.1).

It is constituted by 4 main elements:

1. the support (coloured with light grey), which is fixed and connected to the ground.
2. the driving system (coloured with green), constituted by 2 brushless motors, each actuating a joint.
3. the transmission (coloured with dark grey), which changes the torque and the speed supplied by the motor to the ones requested at joints.
4. the manipulator (coloured with light blue), machine consisting in 4 connected by five revolutionary joints.

The position \( \mathbf{x} = [x; y]^T \) of the joint C can be expressed as function of the actuated joints coordinates \( \mathbf{q} = [\theta^1; \theta^2]^T \).

Figure 2 shows a developed manipulator prototype, whose main feature is the opportunity of changing the distance between the joints connected to the ground \((O_1, O_2)\) through two sliders. The actual configuration allows the two joints to be coincident. This configuration allows to have the wider workspace which is a circle with its center in the origin \((O_1=O_2)\) and radius \( R = L_1+L_2 \).

The graph depicted in Fig.3 highlights the trend of the inverse of the jacobian matrix conditioning number for the configuration described, inside the half workspace.

It is noted that the locus of points where the manipulator is in an isotropic configuration is a circumference. Isotropic behaviour depends only on the distance of the end effector from the origin, while it does not depends on the direction.

Figure 4 is related to a robot configuration corresponding to non coincident joints position. Workspace is reduced and the manipulator behaviour is no more radial symmetric.
5. THE GENERALIZED JACOBIAN MATRIX

Usually in the kinematics optimization the effect introduced by the behaviour of the drive system is not considered. One of the more frequent cases is when actuators are not identical (i.e. different maximum velocities, different maximum torques, etc.). In this case, instead of analyzing the matrix $J^T J$, or its inverse, it will be necessary to consider the generalized Jacobian matrix:

$$ J^* = J D $$  \hspace{1cm} (10) 

where $D$ is a matrix (generally diagonal) to be defined in order to properly weigh the different contributions of $\dot{Q}$ or $F_q$ [6].

Therefore it is essential for the design of the kinematics of a robot, to evaluate the performance indices previously presented to the matrix $DJ$ rather than $J$.

The effect described by the $D$ matrix can be better explained remembering some definitions about isotropy [6]:

1. **geometrical isotropy**, it’s reached when the manipulator, independently by its drive system, is in an isotropic configuration. In this case:
   $$ \text{cond}(J) = 1 \quad \text{or} \quad J^T J = kI $$  \hspace{1cm} (11)

2. **drive system isotropy**, it’s achievable if the behaviour of the drive system is the same in all the configurations reached by the manipulator. It holds:
   $$ \text{cond}(D) = 1 \quad \text{or} \quad D^T D = kI $$  \hspace{1cm} (12)

3. **effective isotropy**, it’s when the robot, driven by a defined drive system, has an isotropic behaviour. In this condition, independently by the condition number of $J$ and $D$, one gets:
   $$ \text{cond}(JD) = 1 \quad \text{or} \quad J^T D^T DJ = kI $$  \hspace{1cm} (13)

**Actuators with different performance**

When the motors used to drive the manipulator are all of the same type (rotational or linear), it is often assumed that all of them have the same maximum performance both in terms of speed and torque (or force).

Conversely, a manipulator can be driven by actuators of the same type, but different from each others in terms of performances.

To visualize this fact, the generalized Jacobian matrix has to be considered instead of the Jacobian one. Matrix $D$ should be defined introducing suitable weights to normalize the performances of the actuators. Such a definition is arbitrary, and there is not a “universal” choice which is suitable for all the situations. In [2] it is suggested to define two matrix, $D_v$ related to velocities and $D_f$ related to forces, as:

$$ D_v = \begin{bmatrix}
\frac{1}{\dot{q}_{1,\text{max}}} & 0 & \cdots \\
0 & \frac{1}{\dot{q}_{2,\text{max}}} & \cdots \\
\vdots & \vdots & \ddots
\end{bmatrix} \hspace{1cm} (14) $$

$$ D_f = \begin{bmatrix}
\frac{1}{f_{1,\text{max}}} & 0 & \cdots \\
0 & \frac{1}{f_{2,\text{max}}} & \cdots \\
\vdots & \vdots & \ddots
\end{bmatrix} \hspace{1cm} (15) $$

where $\dot{q}_{i,\text{max}}$ and $f_{i,\text{max}}$ are respectively the maximum speed achievable by the $i^{th}$ motor and the maximum force deliverable. This choice gives a physical and concrete value to the scaling factors since it depends on the characteristics of the actuators themselves.

Figure 5 shows the manipulator driven by two actuators with different performance ( $\dot{q}_{1,\text{max}} = 1.2 \cdot \dot{q}_{2,\text{max}}$ ).

---

Figure 3 – Evaluation of robot isotropy inside the workspace through the inverse of the Jacobian matrix conditioning number (case $O_1 = O_2$).

Figure 4 – Evaluation of robot isotropy inside the workspace through the inverse of the Jacobian matrix conditioning number (case $O_1 \neq O_2$).
An effect that is usually neglected in the study of manipulator isotropy is due to the presence of transmissions interposed between the structure of the robot and the actuators. Such transmissions change the torques/forces that motors exert on the structure, as well as the speed that they impose.

This effect necessarily changes the kinetostatic properties of the robot and couldn’t be observed by analyzing only the jacobian matrix of the manipulator, but a generalized one should be adopted.

While motors exert torques \( F_a^* \) and speeds \( Q_a^* \), on driven joints are applied forces \( F_\tau = D_v^{-1} F_a^* \) and speeds \( \dot{Q} = D_f \dot{Q}_a^* \) where the matrices \( D_v, D_f \) are defined as:

\[
D_v = \begin{bmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{bmatrix}
\]

(16)

\[
D_f = D_v^{-1} = \begin{bmatrix} 1/\tau_1 & 0 \\ 0 & 1/\tau_2 \end{bmatrix}
\]

(17)

Figure 8 shows a particular of the transmission systems of the considered manipulator. While the two motors are identical, motor \( M_2 \) is connected to a belt transmission to the actuated joint. If \( \tau \neq 1 \) then it results:

\[
\tau_1 \neq \tau_2
\]

(18)

Suppose \( \tau_2 = 2\tau_1 \), the effects on robot isotropy are depicted in Fig. 9,10. For both the cases isotropic behaviour is get dramatically worse with respect to Fig.3,4.

Figure 5 – 5R 2 dof PKM with coincident joints connected to the ground \( (O_1 = O_2) \) driven by two different motors.

Figure 6 – Effects on robot isotropy of actuators with different performance. Evaluation through the inverse of the jacobian matrix conditioning number.

(case \( O_1 = O_2 \)).

Figure 7 – Effects on robot isotropy of actuators with different performance. Evaluation through the inverse of the jacobian matrix conditioning number.

(case \( O_1 \neq O_2 \)).

Transmissions with different transmission ratio

Figure 8 – 5R 2 dof PKM with coincident joints connected to the ground \( (O_1 = O_2) \). A particular of the transmission.
CONCLUSION

Manipulator kinetostatic performances can be analysed through indices related to jacobian matrix especially in terms of isotropy.

Generally this investigation is carried out considering only the geometrical structure of the robot, neglecting the effect of the drive system. This approach may lead to errors and mistakes for same manipulators.

Considering forces or velocities, the Generalized Jacobian Matrix, obtained by identifying appropriate matrices $D_f$ and $D_v$, allow to properly weigh the different contributions of speed and force of each actuator. This operation is performed using some parameters related to the performance of the actuators themselves and therefore gives to the $D$ matrix a physical meaning. Moreover it can describe the possible non-homogeneous behaviour of the drive system that depends on the configuration achieved by the robot.

REFERENCES