Quaternions, Arbitrary Maximally Entangled States, and the Quantization of Two Player, Two Strategy Games

Aden Omar Ahmed
Department of Mathematics
Texas A&M University – Kingsville
Kingsville, Texas 78363-8202 USA
Email: aden.ahmed@tamuk.edu

ABSTRACT

For the quantization of two player, two strategy games by Eisert, Wilkens, and Lewestein, S. Landsburg has constructed a quaternionic representation of the payoff function using which he classified potential Nash equilibria in these games. Landsburg's construction is based on a specific maximally entangled initial state. It turns out, however, that there is an entire class of maximally entangled states any member of which can be used for the "quaternionization" of these games. Here, we present a generalization of Landsburg's construction by using an arbitrary representative from the class of maximally entangled states and classify the potential Nash equilibria in the corresponding two player, two strategy games.

Keywords: Quaternions, Quantum Games, Nash Equilibria.

1. INTRODUCTION

We are concerned with two player, two strategy classical games played under mediated quantum communication ala Eisert, Lewestein, and Wilkins [2]. We discuss our results in the context of the formalism for quantized mixtures given in [1]. In classical mediated communication, players have a referee mediate their game and the communication of their strategic choices. When our players have but two classical pure strategies to choose from, the communication of each player's strategic choices is implemented by the sending of bits to the players, put into an initial state by the referee. Presumably players then send back their individual bits in the other state (Flipped) or in the original state Un-Flipped) to indicate the choice of their second or first classical pure strategy respectively. The returned bits are examined by the referee, who then makes the appropriate payoffs.

When the communication between the referee and the players is over quantum channels, Eisert, Wilkens and Lewestein [2] have proposed families of quantization protocols (Henceforth referred to as EWL protocols) to give players access not just to mere probabilistic mixtures of their pure strategic choices, but also access to quantum superpositions of their original strategic choices. When there are two strategic choices for each player in the classical game, players and the referee communicate over quantum channels via *qubits*, a two pure state quantum system with a fixed observational basis. This observational basis is given in the so-called *Dirac notation* by $|0\rangle$ and $|1\rangle$. This basis also induces an observational basis of the space of the joint states of the players' qubits, denoted for the two player case in the Dirac notation by $|0\rangle \otimes |0\rangle = |00\rangle$, $|0\rangle \otimes |1\rangle = |01\rangle$, $|1\rangle \otimes |0\rangle = |10\rangle$ and $|1\rangle \otimes |1\rangle = |11\rangle$.

Each EWL protocol depends on an initial joint state of the players' qubits prepared by the referee. General actions on a qubit are represented by the elements of the special unitary group *SU(2)* and for our game, the strategic choice represented in classical mediated communication by No Flip is now represented by the identity transformation in *SU(2)*, the strategic choice represented in classical mediated communication by Flip is now represented by an element of the Lie group *SU(2)* which interchanges the pure states of the original observational basis but also maps the initial joint states prepared by the referee under the various profiles of actions of No Flip and Flip to a set of mutually orthogonal joint states. This set of mutually orthogonal joint states forms an alternative observational basis of the joint state space that the referee uses to determine the outcomes, and hence the payoffs for each play of the game. In the two player case these basis elements of the joint state space are written in the Dirac notation by $|NN\rangle$, $|NF\rangle$, $|FN\rangle$ and $|FF\rangle$.

However, upon receipt of their individual qubits, players may choose not just from the matrices representing No Flip and Flip but rather from any element of the Lie group *SU(2)* as one of their pure quantum strategies (i.e. the Q_i's in the formalism of [2]) or even probabilistic combinations therof (i,e. the ΔQ_i's in the formalism of [1]) as their strategic choice and act on their respective qubit accordingly before returning it to the referee. Note that in practice the elements of *SU(2)* here represent quantum superpositions of the players' original two strategies, and the mixed quantum strategies are regular probabilistic combinations of these superpositions. Also note that the players have a vastly broader strategic selection in the pure quantum strategy game G^Q and the mixed quantum strategy game G^{mQ}, even when compared to the already enlarged classical mixed strategy game G^{mix}.

The payoffs to each player of each quantum or mixed quantum strategy profile are computed by the referee by observing the final joint state of the players' qubits with respect to the alternative observational basis of the joint state space described above and the referee then makes the appropriate payoffs. Per the formalism given in [1], this procedure describes for each initial state *I*, a protocol Q_I and a quantized and mixed quantized games G^{Q_I} and G^{mQ_I}.

For two players, if the initial state prepared by the referee is given in the Dirac notation by $|00\rangle$, then the corresponding EWL protocol is not only a complete quantization (see [1]) but is in fact equivalent to the classical game G^{mix}. But when the initial state is given by the maximally entangled state $I = \left(|00\rangle + |11\rangle\right)/\sqrt{2}$, the corresponding EWL protocol still induces a complete quantization of the original game, but is not equivalent to the game G^{mix}, and in contrast to the mixed strategy situation, the corresponding protocols set up onto maps from the appropriate product of the strategy spaces to $\Delta(\mathrm{Im}(G))$, the probability distributions over the image, *Im(G)*, of the payoff function for the game *G*.

The next issue to address is the actual computation of the specific probability distribution over *Im(G)* that arises from a specific profile of players' choices of elements of *SU(2)*, or worse, a profile of players' choices of probability distributions over *SU(2)*. For this task it is useful to employ the *quaternions*, which are a non-commutative, four-dimensional, normed real division algebra with canonical basis consisting of the real number 1 and units *i*, *j*, and *k*. These fundamental units satisfy the so-called *Hamilton relation* $i^2 = j^2 = k^2 = ijk = -1$. This means that each quaternion *q* can be expressed as a linear combination $q = a + bi + cj + dk$ and two such are added or multiplied polynomially, subject to Hamilton's relation above. Each quaternion *q* as above possesses a *quaternionic conjugate* q^* with $q^* = a - bi - cj - dk$.The real-valued multiplicative *norm* (or *length*) on the quaternions is defined by the formula $|q|^2 = q^* q = a^2 + b^2 + c^2 + d^2$ and all non-zero quaternions *q* possess a non-zero inverse $q^{-1} = q^* / |q|$. The *unit* quaternions are those with length 1.

For two player games, by appropriately identifying each player's pure quantum strategies with unit quaternions, S. Landsburg [3] showed that the probability distribution over the outcomes of *G* arising from the profile (*p*, *q*) of quantum strategies in the game G^{Q_I} can be computed directly from the unit quaternion *pq* by merely squaring the real length of each of its canonical components. This description additionally provided the computational capability to calculate the expected payoffs in the game G^{mQ_I} by integrating over the 3-sphere the expression *pq* with respect to the probability distributions over the unit quaternions that form a strategy profile in G^{mQ_I}. This computational capability allowed Landsburg to completely determine the potential Nash equilibria of the games G^{Q_I} and G^{mQ_I}, that is, the game *G* played under the maximally entangled EWL protocol described above.

We present a generalization of Landsburg's construction by using an arbitrary maximally entangled state.

2. GENERALIZING THE LANDSBURG REPRESENTATION

		Player II	
		N	F
I	N	(X_0, Y_0)	(X_3, Y_3)
	F	(X_2, Y_2)	(X_1, Y_1)

Figure 1. A Generic Two Player, Two Strategy Game

In this two player, two strategy game (see Figure 1) we assume the referee initially sends to the two players qubits in the maximally entangled state

$$I_\theta = \left(|00\rangle + e^{i\theta}|11\rangle \right)/\sqrt{2}\,, \tag{1}$$

where θ is a real number. The two classical pure strategies available to the players are No Flip denoted by N, and Flip denoted by F, represented respectively by the $SU(2)$ matrices

$$N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ F = \begin{pmatrix} 0 & \eta \\ -\overline{\eta} & 0 \end{pmatrix} \tag{2}$$

where η is a unit complex number chosen so that the outcome states $|NN\rangle, |NF\rangle, |FN\rangle, |FF\rangle$ of this game form an orthogonal basis of the state space. A direct calculation shows that $\eta^4 = i^2 e^{-2i\theta}$, so setting

$$\eta = \eta_\theta = e^{i(\pi - 2\theta)/4} \tag{3}$$

insures the orthogonality of the outcome states. The players operate on their respective qubits, the first via

$$U_I = \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \tag{4}$$

and the second via

$$U_{II} = \begin{pmatrix} P & Q \\ -\overline{Q} & \overline{P} \end{pmatrix}, \tag{5}$$

respectively. Here A, B, P, and Q are complex numbers with $|A|^2 + |B|^2 = 1$ and $|P|^2 + |Q|^2 = 1$. Ignoring the normalization constant $1/\sqrt{2}$, after the players act, the initial state become with respect to the observational basis

$$\begin{aligned} &\left(AP + e^{i\theta}BQ\right)|00\rangle + \left(-\overline{A}Q + e^{i\theta}B\overline{P}\right)|01\rangle \\ &+ \left(-\overline{B}P + e^{i\theta}\overline{A}Q\right)|10\rangle + \left(\overline{B}\overline{Q} + e^{i\theta}\overline{A}\overline{P}\right)|11\rangle \end{aligned} \tag{6}$$

and

$$\begin{aligned} &\mathrm{Re}\left(AP + e^{i\theta}BQ\right)|NN\rangle + \mathrm{Re}\left(e^{i\pi/4}(e^{-i\theta/2}A\overline{Q} - e^{i\theta/2}B\overline{P})\right)|NF\rangle \\ &- \mathrm{Im}\left(e^{i\pi/4}(e^{-i\theta/2}A\overline{Q} - e^{i\theta/2}B\overline{P})\right)|FN\rangle + \mathrm{Im}\left(AP + e^{i\theta}BQ\right)|FF\rangle \end{aligned} \tag{7}$$

with respect to the alternative observational basis. Hence, up to normalization, the referee observing the game state in the alternative observational basis sees each pure action state with probability given by

$$pr(NN) = \left[\mathrm{Re}(AP + e^{i\theta}BQ)\right]^2 \tag{8}$$

$$pr(NF) = \left[\mathrm{Re}\left(e^{i\pi/4}(e^{-i\theta/2}A\overline{Q} - e^{i\theta/2}B\overline{P})\right)\right]^2 \tag{9}$$

$$pr(NF) = \left[\mathrm{Im}\left(e^{i\pi/4}(e^{-i\theta/2}A\overline{Q} - e^{i\theta/2}B\overline{P})\right)\right]^2 \tag{10}$$

$$pr(NN) = \left[\mathrm{Im}(AP + e^{i\theta}BQ)\right]^2 \tag{11}$$

Now consider the identification of the group SU(2) with the group S^3, considered as the unit quaternions equipped with quaternionic multiplication, via the maps

$$U_I \leftrightarrow p_\theta = A + B\overline{\eta}_\theta e^{i\frac{\theta}{2}} j \tag{12}$$

and

$$U_{II} \leftrightarrow q_\theta = P - \overline{Q}\,\overline{\eta}_\theta e^{-i\frac{\theta}{2}} j\,, \tag{13}$$

where U_I and U_{II} are as given above. It is straightforward to check that p_θ and q_θ as above are unit quaternions and that the maps given in Eq.

(12) and (13) are group isomorphisms for all real number θ.

Now suppose that player 1 chooses the unit quaternion p_θ as defined in Eq. (12) and player 2 chooses the unit quaternion q_θ as defined in Eq. (13). If we write the product $p_\theta q_\theta$ as

$$p_\theta q_\theta = \left(A + B\overline{\eta}_\theta e^{i\frac{\theta}{2}} j\right)\left(P - \overline{Q}\,\overline{\eta}_\theta e^{-i\frac{\theta}{2}} j\right)$$
$$= \pi_0(p_\theta q_\theta)\cdot 1 + \pi_1(p_\theta q_\theta)\cdot i + \pi_2(p_\theta q_\theta)\cdot e^{i\frac{\theta}{2}} j + \pi_3(p_\theta q_\theta)\cdot e^{i\frac{\theta}{2}} k,$$

where the $\pi_t(p_\theta q_\theta)$ are real numbers, then we are led to the following theorem

Theorem 2.1. *If in the maximally entangled two player, two strategy quantum game, player 1 plays the pure quantum strategy U_I and player 2 the pure quantum strategy U_{II}, then the probability distribution over the set of outcomes NN, NF, FN, and FF is given by*

$$pr(NN) = [\pi_0(p_\theta q_\theta)]^2,\ pr(NF) = [\pi_3(p_\theta q_\theta)]^2$$
$$pr(FN) = [\pi_2(p_\theta q_\theta)]^2,\ pr(FF) = [\pi_1(p_\theta q_\theta)]^2$$

The proof is straightforward calculations. Note that if the players employ unit quaternions given with respect to the standard basis $B = \{\ 1, i,\ j, k\ \}$, then the probability distributions given above become

$$pr(NN) = \left[\pi_0\left(M_\theta^{-1}(pq)\right)\right]^2,\ pr(NF) = \left[\pi_3\left(M_\theta^{-1}(pq)\right)\right]^2$$
$$pr(FN) = \left[\pi_2\left(M_\theta^{-1}(pq)\right)\right]^2,\ pr(FF) = \left[\pi_1\left(M_\theta^{-1}(pq)\right)\right]^2$$

where M_θ is the basis change matrix of the basis $B_\theta = \left\{\ 1, i, e^{i\frac{\theta}{2}} j, e^{i\frac{\theta}{2}} k\ \right\}$ to the standard basis *B*. This result leads to the following definition:

Definition 2.2. *Let G be the game depicted in Figure 1. Then the associated quantum game $G^{Q_{I\theta}}$ is the two player game in which each player's strategy space is the unit quaternions, and the payoff functions for players 1 and 2 are defined as follows:*

$$P_I(p,q) = \sum_{t=0}^{t=3}\left[\pi_t\left(M_\theta^{-1}(pq)\right)\right]^2 X_t \qquad (14)$$

$$P_{II}(p,q) = \sum_{t=0}^{t=3}\left[\pi_t\left(M_\theta^{-1}(pq)\right)\right]^2 Y_t \qquad (15)$$

Note that one can recover Landsburg's representation by setting $\theta = 0$.

3. NASH EQUILIBRIUM

It is straightforward calculations to show that the mixed quantum strategies

$$\mu = \frac{1}{4}(1 + i + j + k) \qquad (16)$$

and

$$\nu = \frac{1}{4}(1 + i + j + k) \qquad (17)$$

for players 1 and 2, respectively, are best replies to each other thereby giving a Nash equilibrium in $G^{mQ_{I\theta}}$ for all real numbers θ. This equilibrium yields to the players expected payoffs that are the average of the classical individual payoffs for the players.

4. CONCLUSIONS

We extended Landsburg's representation to two player, two strategy games where the initial state is chosen arbitrarily from a class of maximally entangled states with equal superpositions.

5. REFERENCES

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