Teaching Geometry and Research on Fractal Tilings

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Abstract

In this paper, we demonstrate how our research findings regarding fractal tilings arose through the teaching of a geometry course, and describe the mathematical relationships between the content of the geometry course and our research results. We share this delightful experience and demonstrate how research can be closely related to teaching activities in university.

Keywords: Geometry Course, Linking Teaching and Research, Fractals, Fractal Tilings, Maple, Reptile.

1. Introduction

The geometry course at the National Kaohsiung Normal University covers Euclidean and fractal geometry. Because these students will eventually become teachers in high school, a number of interesting subtopics of Euclidean and fractal geometry are also taught, including tilings, polyhedra, the Golden ratio, and fractals. Students are provided the opportunity to explore the properties of geometry using dynamic geometry software, such as GSP and GeoGebra, as well as computer algebra systems, including Maple. After teaching the above content for two years, the author sought to expand the curricula with new materials related to tilings using fractal shapes. This opened a door to the newly develop field of fractal tilings.

The internet provides many sources of information related to the topic of fractal tilings; however, this new field is scarcely touched upon in textbooks on geometry or fractals. The research results of the author include a rigorous definition of fractal tilings in [1] and an algorithm with which to generate self-similar forms in [2]. The exploration of research questions in this geometry course has proven highly rewarding, both for instructors and students, particularly with regard to the invention of new fractal shapes using Maple by students. In this paper, we share this delightful experience and demonstrate how research can be closely related to teaching activities in university.

About the relationship between research and teaching, in [3,4] McNay, Healy et al. indicated the trend of separation of research and teaching in university "Structural changes: research centres housed staff freed from teaching responsibilities; graduate schools became the arenas for research, leaving department to organize undergraduate teaching. Each of these [developments] was particular and peculiar, but the trend was gradually of a separation, structurally of research from teaching".

More recently, one of the topics to study linking teaching and research is called "The Research informed Teaching (RiT)" introduced in the homepage of Staffordshire University in United Kindom as "...recognise the importance of the reciprocal relationship between teaching and research in enhancing the students' learning experience In the University Research and Enterprise Strategy, for example, it is stated that applied research activity will underpin the student experience through its support of learning and teaching, and many staff already make good use of their research in the teaching context ." It shows that the trend of linking research and teaching is obvious.

In [5,6,7] the typology of teaching-research links was developed as 1.Teaching can be research-led 2. Teaching can be research-oriented 3. Teaching can be research-based 4. Teaching can be research-informed. In the geometry course, we conducted students to understand the processes by which fractal knowledge is produced and learn how to simulate fractals by Maple programming. So our course should fall on the second category of this typology. It was the teaching experience in this course that made the author invent new results about the fusion topic of tilings and fractals. The remainder of this paper is organized as follows: Section2 describes the main ideas of tilings and fractals, and the computer simulation of fractals with Maple by students. Section 3 describes our published research results about fractal tilings. And the conclusion is given in Section 4.

2. Geometry Course Teaching

The geometry course is taught three hours per week for two semesters. The first part of the second semester is devoted to the geometry of tilings and fractal geometry is covered in the second. The need to make these topics comprehensible to students led the author to develop a novel fusion of tilings and fractals. In the following subsections, we outline the main ideas of these two topics in detail.

Tilings

We review here some definitions of tilings in the book by Grünbaum and Shephard [8].

Definition 1. A *plane tiling* T is a countable family of closed topological disks $\mathrm{T} = \{T_1, T_2, \cdots\}$, which cover the plane without gaps or overlaps. More explicitly, the union of the sets $T_1, T_2, \cdots$ (which are known as the tiles of T) spans the entire plane, and the interior of the sets T_i are pairwise disjoint. A closed topological disk means a set homeomorphic to a closed circular disk.

Definition 2. A *monohedral tiling* T is a tiling whose tiles are all the same size and shape, i.e.,

$$T_1 \overset{congruent}{\cong} T_2 \overset{congruent}{\cong} \cdots .$$

Fractals with Maple Simulation

We see in the sequel that students presented their Maple codes to simulate their own designed fractals in the computer room.

At first we need review some notations and terminology for fractals. As an example, if we want to generate a von Koch curve we assign proper mappings $\{\phi_1, \phi_2, \cdots, \phi_n\}$ to constitute an iterated function system (IFS) [9]. We define the initiator and the generator as in [10].

We denote $H(A,k)$ to be a scaling with center A and scaling ratio k , that is, $H(A,k)(B) = C$ means that A, B, C are collinear and $\frac{AC}{AB} = k$. $T_{AB}(C)$ is defined to be a translation so that if $T_{AB}(C) = D$ then $CD // AB, CD = AB$. $R(O,\theta)$ is a rotation with center O and angle θ.

Let $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ be such that:

$$\varphi_1 = H(A, \frac{1}{3}),$$

$$\phi_2 = R(C, \frac{\pi}{3}) \circ T_{AC} \circ H(A, \frac{1}{3}),$$

$$\phi_3 = R(E, -\frac{\pi}{3}) \circ T_{AC} \circ H(A, \frac{1}{3}),$$

$$\phi_4 = T_{AE} \circ H(A, \frac{1}{3}),$$

and define

$\Phi(B) := \bigcup_{i=1}^{n} \phi_i(B)$. Then the von Koch curve is: $A = \lim_{n \to \infty} \Phi^{\circ n}(U)$, see Fig. 1.

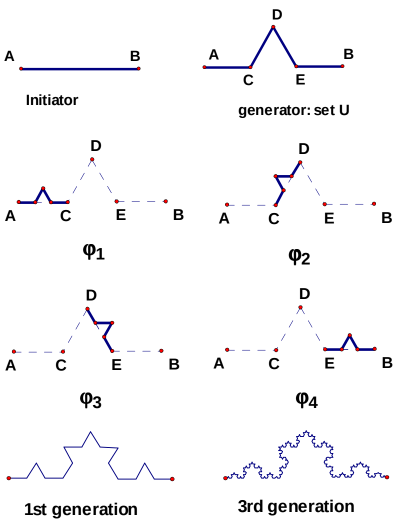


Fig. 1

We showed at first the Maple codes of Sierpinski carpet in [11] to students then explain the meaning of the codes in brief to them in computer room.

```
> restart;
> serp3:=proc(L::algebraic, lev::integer,x0::algebraic,
y0::algebraic)
local i,j;
global p,s;
options remember;
if s=0 then
p[0]:=plots[polygonplot]([[x0,y0],[x0,y0+L],[x0+L,y0
+L],[x0+L,y0]],style=line):
fi;
s:=s+1;
p[s]:=plots[polygonplot]([[x0+L/3,y0+L/3],[x0+L/3,y
0+2*L/3],[x0+2*L/3,y0+2*L/3],[x0+2*L/3,y0+L/3]],c
olor=blue):
if lev >1 then
  for i from 0 to 2 do
     for j from 0 to 2 do
        if abs(i-1)+abs(j-1) >0 then
          serp3(L/3,lev-1,x0+i*L/3,y0+j*L/3);
        fi;
     od;
  od;
fi;
RETURN(plots[display]([seq(p[i],i=0..s)],scaling=con
strained))
# In fact s=(8^lev-1)/7.
end:
> s:=0;serp3(100,3,0,0);
```

$$s := 0$$

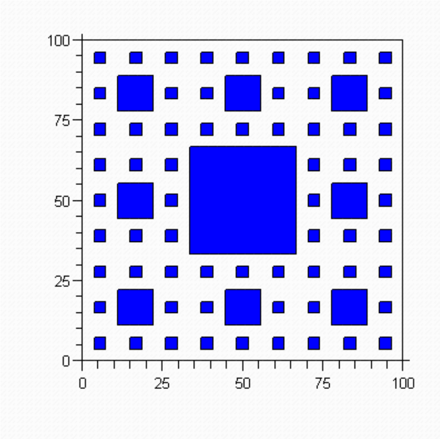


Fig. 2

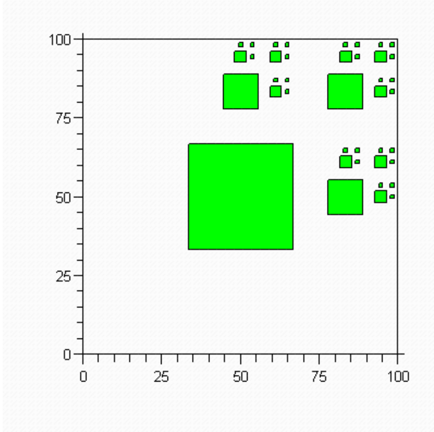


Fig. 3

Fig.2 is the running result graph of the above maple codes..

Then we showed the variant codes to get modified Sierpinski carpet as in the fig. 3 to let the students know more clearly how the codes work.

```
> serp2:=proc(L::algebraic, lev::integer,x0::algebraic,
y0::algebraic)
local i,j;
global p,s;
options remember;
if s=0 then
p[0]:=plots[polygonplot]([[x0,y0],[x0,y0+L],[x0+L,y0
+L],[x0+L,y0]],style=line):
fi;
s:=s+1;
p[s]:=plots[polygonplot]([[x0+L/3,y0+L/3],[x0+L/3,y
0+2*L/3],[x0+2*L/3,y0+2*L/3],[x0+2*L/3,y0+L/3]],c
olor=green):
if lev >1 then
   for i from 1 to 2 do
      for j from 1 to 2 do
         if abs(i-1)+abs(j-1) >0 then
            serp2(L/3,lev-1,x0+i*L/3,y0+j*L/3);
         fi;
      od;
   od;
fi;
RETURN(plots[display]([seq(p[i],i=0..(3^lev-
1)/2)],scaling=constrained))
end:
> s:=0;serp2(100,4,0,0);
```

$$s := 0$$

Then we assigned homework to all student teams to design (invent) and simulate their fractals with Maple programming. We found the students could design their fractal shape by modifying the codes in [11] and by repeating computer simulating. The following Fig. 4-Fig. 7 are part of photos of students while presenting their homework in computer room.

Fig. 4 One team was presenting their Maple codes and designed fractal in computer room.

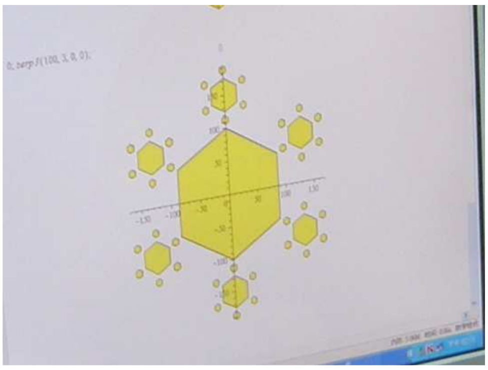


Fig. 5 The detail of the designed fractal in the above figure.

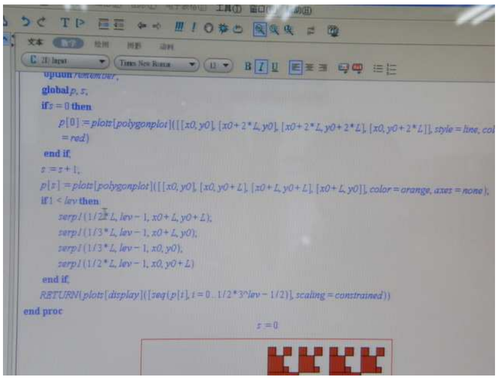


Fig. 6 The Maple codes of a designed fractal by another team.


Fig. 7 Another team was preparing the presentation.

3. Research Results

After teaching the above topics about fractals and tilings in Geometry course, the author presented some research results about fractal tilings in [1,2]. We state the results in the following in brief.

In [12] they suggested a way of generating a fractal tiling; they give the following intuitive definition for a *fractal tile*.

Definition 3. A *fractal tile* is a tile whose boundary is composed of fractal curves, or say, a topological closed disk with fractal boundaries.

In [1] we give a rigorous definition of fractal tilings. At first we review a prefractal [13].

Definition 4. A *prefractal* is the intermediate shape for generating a fractal using the IFS method.

Then we give the definition of prefractal tilings.

Definition 5. Given a tile with fractal boundary , we denote the tile by T and its fractal boundary by Γ. We define the (*kth*) *prefractal tile of* T to be a set (tile) with (*kth*) prefractal of Γ as boundary.

The following is a rigorous definition of fractal tiling presented by the author in [1].

Definition 6. A tile with fractal boundary can tile the plane if its every k th prefractal tile can tile the plane for every $k \in N$. If so, we call this tile with fractal boundary *a fractal tile*.

Another research results in [1] included five methods, including Escher's tiling pictures methods and the Conway criterion to create the fractal tilings. By the results in [1] it is easy to check the well-known fractal fudgeflake to be a fractal tilings see Fig 8. .

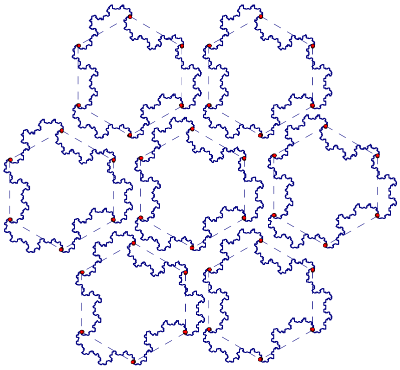


Fig. 8 The fudgeflake as a fractal tilings

Our another research result is an algorithm to generate fractal reptiles [2].
Definition 7. A *k-rep tile* is defined as any set T that can be dissected into k congruent parts T' each of which is similar to T . If the above T is disk-like, then we say that T is a *disk-like k-rep tile.* A fractal tile is called *a fractal k-reptile* if it is a k-reptile.

Sphinx is a famous example of a 4-rep tile. It is well known that the Sierpinski gasket constitutes of 3^n copies of small congruent one for any $n \in \mathbb{N}$. So the Sierpinski gasket is a non-disk-like 3^n -rep tile for any $n \in \mathbb{N}$.

We investigate a well known process to generate Gosper snowflake (Gosper island). With the aid of Escher-style rules we can realize more deeply why this process works. The investigation proceeds as follows. The 0th step we begin with a hexagon G_0 (Fig. 9). We can call it the initiator. Step 1. we use 7 congruent hexagons scaled from G_0 to form the new polygon called G_1. For convenience we denote the scaled one also by G_0 in the figure. We can see that because G_1 is modified from a hexagon by the Escher parallel translation rule, it follows that G_1 can tile the plane well. We need the dashed hexagon attached to G_1 to indicate the relative position. We call it the *referenced hexagon of* G_1 (Fig. 10). Step 2. shrink G_1 as small as the scaled G_0 in G_1 and replace every small G_0 by scaled G_1 by coinciding the *referenced hexagon of* G_1 with small G_0 to form G_1. Here we can see that G_2 is modified from G_1 by the Escher parallel translation rule. So we know that G_2 can tile the plane as well (Fig. 11). Step 3 Similarly, we replace every small G_1 in G_2 by scaled G_2 to get G_3. Continue this process indefinitely to get the convergent shape which is the Gosper snowflake.

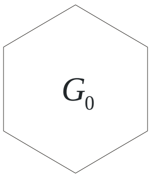
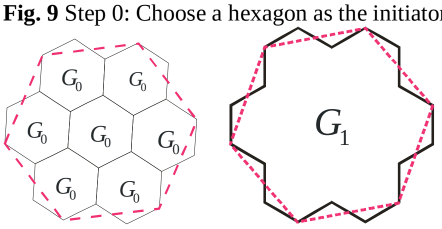


Fig. 9 Step 0: Choose a hexagon as the initiator



Step1: Find a generator which is satisfied with Escher rules.

Fig. 10 Find suitable initiator and generator of Gosper snowflake.

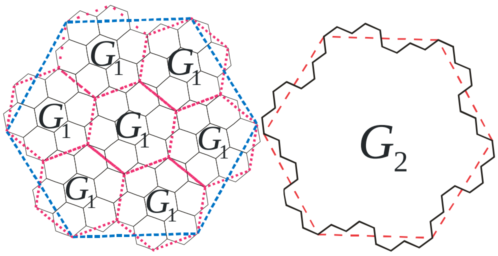


Fig. 11 Step 2: Replacing G_0 with G_1 to get G_2.

With the aid of the above investigation, we can provide an informal proof as follows. Let $G_{n,k}$ denote the scaled previous generation shapes in G_{n+1}. Because $G_{n+1} = \bigcup_{k=1,\cdots,7} G_{n,k}$ for every $n \in \mathrm{N}$, with no overlap and no gap. Taking limit on both sides to get

Gosper= $\lim_{n\to\infty} G_{n+1} = \lim_{n\to\infty} \bigcup_{k=1,\cdots,7} G_{n,k}$

$= \bigcup_{k=1,\cdots,7} \lim_{n\to\infty} G_{n,k} = \bigcup_{k=1,\cdots,7}$ small Gosper

For the no overlap and no gap union we get that the Gosper snowflake is a 7-reptile.

Base on the above investigation and Escher-style rules we can now present an algorithm to generate fractal reptiles. We find the key point is a criterion with which the polygon initiator P_0 and the polygon generator P_1 must satisfy.

Criterion 1. Several congruent scaled-down P_0 can be joined together to form P_1, i.e., $P_1 = \bigcup_{k=1}^{m} P_{0k}$ where P_{0k} is congruent to a scaled-down P_0, such that P_1 is modified from P_0 according to some Escher-style rule or Conway criteria.

Algorithm for generating fractal reptiles:

Step 1. Choose a polygon P_0 (initiator) and a polygon generator P_1 which satisfy criterion 1. Then we should attach a referenced P_0 on P_1 (as stated in the above Gosper example) to proceed the next step.

Step 2. Replacing every P_{0k} in P_1 by scaled-down P_1 to form P_2. The Escher-style rule promises that the union is no overlap and no gap. Also we need to attach a referenced P_0 on P_2 for the successive replacing process.

Step 3. Continue similar way in Step2 indefinitely to get a limit set which is a fractal m-reptile.

4. Conclusion

In this paper, we demonstrate how our research findings regarding fractal tilings arose through the teaching of a geometry course, and describe the mathematical relationships between the content of the geometry course and our research results. This experience is a poignant demonstration of the close relationship between research and teaching.

5. Acknowledgement

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