Option Pricing by Fuzzy Logic Based Signal Processing

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Abstract—One of the best examples of mathematically rigorous signal processing in finance is the Black-Scholes model for price evolution of financial options. To address the same problem, this paper proposes a Takagi-Sugeno-Kang (TSK) fuzzy rule-based option pricing model that requires only a small number of rules to describe highly complex financial signals such as option prices. The findings for this data-driven approach indicate that the TSK model presents a robust option pricing tool that is superior to an array of well-known parametric models from the literature, including the Black-Scholes model. In addition, its predictive performance is consistently no worse than that of a non-parametric feedforward neural network model.

1. INTRODUCTION

Options are financial contracts or the so-called financial derivatives that “derive” their value (i.e., their price) from the value of the underlying security that can include goods, stocks, exchange rates, interest rates and stock market indices. For example, a call (put) option based on a stock provides its buyer with a right to purchase (sell) a predetermined amount of stocks at a contracted price (“strike price”) on or by a specific date (“maturity”). For this right the buyer of an option pays a price called the premium. In general, options are often referred to as plain vanilla derivatives because their price is determined by the underlying, which is in the current paper the S&P 500 stock market index. Call options are more profitable for the buyer when, ceteris paribus, the price of the underlying (S) increases or the strike price (K) decreases. Therefore, intuitively, these two variables must be integral parts of the option pricing formula. Further, when time to maturity (\(\tau\)) increases, call options become more valuable. This is explained by the fact that it is more likely that the option will be “in the money” (\(S_t-K>0\)) and, thus, worthwhile exercising. Option contracts typically undergo large price swings until maturity, which makes it very challenging to understand how they are priced.

One of the best examples of mathematically rigorous signal processing in finance is the Black-Scholes model [1] for price evolution of financial options [2]. They proposed the first theoretical framework for option pricing, which was almost immediately accepted and celebrated by both academics and practitioners. However, this model relies on a number of restrictive assumptions that have often been blamed for certain pricing errors of the Black-Scholes formula. For instance, Bakshi et al. [3] document pricing biases for put and call options, which are referred to as the “volatility smile”. The observed biases have motivated the development of more complex non-parametric option pricing models such as artificial neural networks.1

Non-parametric methods are generally based on a trade-off between smoothness and goodness-of-fit. This trade-off is usually controlled by the choice of a parameter in the estimation procedure, which is a difficult task. It may result in the lack of stability that is detrimental to the out-of-sample performance of non-parametric methods. This is the key reason why one may prefer a parsimonious parametric model. This paper proposes a robust parametric Takagi-Sugeno-Kang (TSK) fuzzy rule-based option pricing model that requires only a small number of rules to describe highly complex financial signals such as option prices. As a parametric system, it does not suffer from the lack of parameter transparency, and in addition, it allows the researcher to control the trade-off between smoothness and goodness-of-fit through a single parameter, the number of clusters.

2. TSK MODEL’S DESIGN

The basic idea of the TSK model [5] is the fact that an arbitrarily complex model explaining a financial signal is a combination of mutually interlinked sub-models that process certain aspects of a financial signal of interest (in other words, each price signal is classified according to the S, K and \(\tau\) signals). If \(C\) regions, each corresponding to an individual sub-model, are determined in the state-space under consideration, then the behavior of the system in these regions can be described by the simpler functional forms. If the sub-model is assumed to be linear and one rule is assigned to each sub-model, the TSK fuzzy model can be represented with \(C\) rules of the following form:

\[
R_i: \quad \text{if } x_1 \text{ is } A_{1i} \text{ and } x_2 \text{ is } A_{2i} \text{ and } \ldots \text{ and } x_n \text{ is } A_{ni} \text{ then } y_i = a_{i0} + b_i, \quad i = 1, 2, 3, \ldots, C,
\]

where \(R_i\) is the \(i\)-th rule, \(x_1, x_2, \ldots, x_n\) are the input variables, \(A_{1i}, A_{2i}, \ldots, A_{ni}\) are the fuzzy sets assigned to corresponding input variable, the variable \(y_i\) represents

1 Excellent review of the option pricing literature can be found in Garcia et al. [4].
the value of the $i$-th rule output, and $a_i$ and $b_i$ are parameters of the consequent function. The final output of the TSK fuzzy model for arbitrary inputs $x_k$ can be calculated as a weighted average of the rule consequents:

$$\hat{y}_k = \frac{\sum_{i=1}^{C} [\beta_i(x_k) x_k]}{\sum_{i=1}^{C} \beta_i(x_k)}, \quad k = 1, 2, 3, ..., N,$$

where $\beta_i$ is the weight assigned to the $i$-th rule.

3. DATA

The data are provided by DeltaNeutral and represent the daily S&P 500 index European call option prices, taken from the Chicago Board Options Exchange. Call options across different strike prices and maturities are considered for the 1987-1993 period. Since it is one of the deepest and the most liquid option markets in the United States, the S&P 500 index option market is sufficiently close to the theoretical setting of the Black-Scholes model.

Options with zero volume are not used in the estimation. As in [6], the data for each year are divided into three parts: first two quarters (estimation data), third quarter (validation data) and fourth quarter (testing data). This produced the following non-overlapping subsamples:

- **1987:** Training data: 3610, Validation data: 2010, Testing data: 2239 – observations
- **1988:** Training data: 3434, Validation data: 1642, Testing data: 1479 – observations
- **1989:** Training data: 3052, Validation data: 1565, Testing data: 1515 – observations
- **1990:** Training data: 3610, Validation data: 2010, Testing data: 2239 – observations
- **1991:** Training data: 4481, Validation data: 1922, Testing data: 2061 – observations
- **1992:** Training data: 4374, Validation data: 1922, Testing data: 1848 – observations
- **1993:** Training data: 4214, Validation data: 1973, Testing data: 2239 – observations

The optimal number of clusters ($K$) is determined from the out-of-sample performance on the validation set with respect to the mean-squared prediction error (MSPE) or the mean-absolute percentage error (MAPE). The predictive performance on the testing data is finally assessed with the MSPE and the MAPE criteria for each year.

The out-of-sample pricing performance of the TSK model is first compared to the well-known benchmark – the Black-Scholes model. The Black-Scholes call prices ($C_t$) are computed using the standard formula.

The statistical significance of the difference in the out-of-sample (testing set) performance of alternative models is tested using the Diebold-Mariano test [9]. We test the null hypothesis that there is no difference in the MSPE of the two alternative models. The Diebold-Mariano test statistic for the equivalence of forecast errors is given by

$$DM = \frac{1}{M} \sum_{i=1}^{M} d_i,$$

where $M$ is the testing set size and $f(0)$ is the spectral density of $d_i$ (the forecast error is defined as the difference between the actual and the forecasted output value) at frequency zero. Diebold and Mariano show that $DM$ is asymptotically distributed in a $\mathcal{N}(0,1)$ distribution.

4. OPTION PRICING MODEL

The option pricing formula for the S&P 500 index call options is defined as in Garcia and Gençay [6]:

$$C_t = \phi(S_t, K, \tau),$$

where $C_t$ is the call option price, $S_t$ is the price of the underlying asset, $K$ is the strike price, and $\tau$ is the time to maturity (number of days). Assuming the homogeneity of degree one of the pricing function $\phi$ with respect to $S_t$ and $K$, one can write the option pricing function as follows:

$$\frac{C_t}{K} = \frac{f(S_t, K, \tau_2)}{f(S_t, K, \tau_1)} = \phi(x_1, x_2)$$

Therefore, the model has only two inputs, and it can be argued that other variables such as volatility (standard deviation of the underlying asset), risk-free interest rate, and dividends paid on the underlying asset should be included. As the focus of the paper is to introduce the TSK model as a new option pricing methodology, these inputs will not be included.\(^2\)

In addition to the Black-Scholes model, the TSK model will be compared to the non-parametric neural network (NN) model with the hint from Garcia and Gençay [6] and also to the semi-parametric estimator by Aït-Sahalia and Lo [8]. The NN model with the “hint” involves using additional prior information about the properties of an unknown pricing function to guide the learning process. This means breaking up the pricing function into two parts, one controlled by $S_t/K$ and the other one by a function of $\tau$. Each part contains a cumulative distribution function, which is estimated

\(^2\) In fact, we added the volatility and the interest rate as well as the implied volatility and the implied interest rate, and these inputs were informative for the models. The results are available upon request.
nonparametrically through feedforward NN models. Garcia and Gençay [6] show that the NN model with the hint dominates the standard backpropagation NN model. Recall that according to Gençay and Gibson [7], the NN option pricing model outperforms standard parametric approaches (SVJ, SI and SV) for the S&P 500 stock market index in a robust fashion. Hence, achieving an out-of-sample performance that is comparable to the NN model with the hint using a parametric model appears to be a challenging task. However, the TSK model presents a parametric model that has a unique “divide-and-conquer” property. This property makes it distinct from the standard parametric option pricing models, which are mainly focused on relaxing the Black-Scholes assumptions with regard to volatility and interest rates.

5. RESULTS

The out-of-sample pricing performance of the TSK model is evaluated on the last quarter of each sample year by comparison with the neural network [7], the Black-Scholes model, and the semi-parametric model by Aït-Sahalia and Lo [8] (Table 1). The NN model with the hint is estimated using the early stopping technique. To control for potential data snooping biases, as in [6], the estimation is repeated ten times from ten different sets of starting values and the average MSPEs are reported in the first and in the second columns, respectively. For the AL model, the MSPE across all years is reported for the bandwidth that yields the smallest MSPE on the validation sample. It is worth mentioning that the out-of-sample performance of this model could not be improved by using a range of other bandwidth values. However, although in most of the cases it underperforms even the Black-Scholes model, the AL model exhibits superior performance in 1987.

Overall, the most accurate model is difficult to determine because, out of seven years in the sample, the NN with the hint model is statistically superior in 1988 and 1991, while the TSK model is dominant in 1987 and 1993. According to the DM statistic, neither of the two models is statistically more accurate in the remaining three years. Interpreting the results in 1987 requires care since it is the year of the stock market crash. By investigating this issue further, we observe that the inaccuracy of the NN with the hint and the TSK models in 1987 is due to their mispricing out-of-the-money calls, which does not happen in any of the “normal” years. In general, it can be concluded that the models become less reliable when the markets are more volatile.

Next, we focus on the two best performing models – the NN with the hint and the TSK models -- by comparing their out-of-sample MAPEs. Table 2 compares the average MAPE of the NN model with the hint to that of the TSK model across the years. It can be observed that the TSK model demonstrates superior performance in most years, except for 1989 and 1990, when the MAPE figures are roughly equal. Therefore, based on the results for both error measures, we conclude that the TSK model does not perform worse than the non-parametric NN with the hint model, and is even dominant in the case of MAPE.

| 1987  | 36.20 | 16.06 |
| 1988  | 3.94  | 2.78  |
| 1989  | 1.02  | 1.02  |
| 1990  | 1.84  | 1.87  |
| 1991  | 1.83  | 1.72  |
| 1992  | 1.34  | 0.96  |
| 1993  | 2.06  | 1.48  |

Notes: This table reports the out-of-sample average mean-absolute percentage error (MAPE) of Garcia and Gençay’s [6] feedforward neural network model with the hint (NN with hint) and the TSK model. The average MAPE figures are presented as averages across ten different random training seeds.

| 1987  | 16.7  | 6.3829 | 3.3665 | 4.38  | -8.49  |
| 1988  | 0.7114| 0.8086 | 6.0585 | 2.07  | 7.68   |
| 1989  | 0.4138| 0.4237 | 2.3029 | 1.42  | 0.69   |
| 1990  | 0.6761| 0.6797 | 2.4391 | 2.62  | 0.10   |
| 1991  | 0.3498| 0.4022 | 13.8525| 1.73  | 8.27   |
| 1992  | 0.1511| 0.1559 | 0.3625 | 1.36  | 0.05   |
| 1993  | 0.1054| 0.0570 | 2.9631 | 0.74  | -7.64  |

Notes: This table reports the out-of-sample average mean-squared prediction error (MSPE) of Garcia and Gençay’s (2000) feedforward neural network model with the hint (NN with hint) and the TSK model. The table also reports the out-of-sample MSPE of Aït-Sahalia and Lo’s (1998) non-parametric kernel estimator (AL) and the Black-Scholes model (BS model). The average MSPEs for the NN with hint model and the TSK model are presented as averages across ten different random training seeds. DM denotes the Diebold and Mariano (1995) test statistic. This test is used to assess the statistical significance of the forecast gains of the NN with the hint model relative to the TSK model. All MSPE figures have been multiplied by 10^4.
It is essential to understand where the pricing accuracy of the TSK model comes from, i.e., for which option types it exhibits outstanding pricing accuracy. For this purpose, we divide each of the moneyness \((S/K)\) and the time to maturity \((\tau)\) ranges into three non-overlapping intervals, which gives us nine types of options, as follows:\(^3\)

- **Type 1**: Out-of-the-money, short term options: \((S/K) < 0.97\) and \(\tau < 0.1\);
- **Type 2**: Near-the-money, short term options: \(0.97 \leq (S/K) \leq 1.05\) and \(\tau < 0.1\);
- **Type 3**: In-the-money, short term options: \((S/K) > 1.05\) and \(\tau < 0.1\);
- **Type 4**: Out-of-the-money, medium term options: \((S/K) < 0.97\) and \(0.1 \leq \tau \leq 0.2\);
- **Type 5**: Near-the-money, medium term options: \(0.97 \leq (S/K) \leq 1.05\) and \(0.1 \leq \tau \leq 0.2\);
- **Type 6**: In-the-money, medium term options: \((S/K) > 1.05\) and \(0.1 \leq \tau \leq 0.2\);
- **Type 7**: Out-of-the-money, long term options: \((S/K) < 0.97\) and \(\tau > 0.2\);
- **Type 8**: Near-the-money, long term options: \(0.97 \leq (S/K) \leq 1.05\) and \(\tau > 0.2\);
- **Type 9**: In-the-money, long term options: \((S/K) > 1.05\) and \(\tau > 0.2\).

Figure 1 provides more detailed results for all nine option types. In each of the nine panels, we plot the squared difference between the actual option price and the price estimated by the TSK model: \((\hat{c}_t - c_t)^2\), where \(t=1,...,M\) (size of the testing set, which varies by the option type), for the 1993 data. Similar plots can be constructed for each of the other years. Clearly, the pricing biases that arise from time to maturity are more serious than the ones related to the moneyness of the options. In other words, when moving from left to right in Figure 2, the fluctuations in the pricing error are more pronounced than when moving from the top to the bottom panels. These findings are similar to the ones for the feedforward NN model in [7].

6. Conclusions

This paper introduces a parsimonious, parametric TSK model that can to a certain extent account for the non-normality in S&P 500 index returns. Moreover, this model demonstrates option pricing performance similar to that of non-parametric NN models. We also show the TSK approach presents a robust option pricing tool, superior to an array of well-known parametric models from the literature. We attribute the success of the TSK model to its “divide-and-conquer” ability to decompose the option pricing model into a number of sub-models across moneyness and maturity ranges. Unlike its non-parametric counterparts, the TSK model is not susceptible to parameter instability or overfitting. Specifically, during the NN training phase, various optimization methods such as the Levenberg-Marquardt or the Gauss-Newton algorithms are frequently used. In contrast to the LS method applied for TSK estimation, these methods are very computationally demanding and prone to the above-mentioned problems. The LS method is much faster, and thus more adequate for on-line option pricing applications that involve frequent trading decisions. In addition, the partitioning of the input-output space of each model is carried out using a fuzzy clustering method. This makes the identification of the cluster centers more efficient, thus potentially arriving at models with a small number of rules and a greater degree of interpretability.

The empirical evidence indicates that the “divide-and-conquer” TSK approach can be very useful for approximating option functions in the presence of heterogeneous data. For example, an extremely volatile data region may be followed by a relatively stable data sequence. In the case of the NN approach (or any other non-parametric method), this situation can make it challenging for the model to extract information and generalize the global option function. We demonstrate how the TSK model overcomes these challenges across various options types. However, in line with the findings for other option pricing methods, the TSK model exhibits certain biases when used for pricing longer term options.

There are several possible extensions to the current approach. The fuzzy clustering component of the model could be complemented by more advanced methodologies from statistical learning theory, such as

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3 The boundaries between the ranges are adapted from Garcia and Gençay [6].
genetic algorithms [10]. It is worth noting that the ability to differentiate among various option types or “data regimes” (i.e., clusters of data) represents one of the crucial future research directions. In the present paper, this process is based on a two-dimensional input space. However, additional inputs such as the interest rate and volatility can be provided. To ensure superior pricing performance in the presence of four inputs, the method would probably need to be compared to some alternative clustering algorithms.

The results and research questions addressed by the TSK model can also be revisited using data from individual stock options, American-type options and currency and commodity options. The robustness and accuracy of the TSK model presented in this paper can be best judged by examining its pricing properties with regard to other types of options. There is certainly room to explore all these issues in future research.

Figure 2. Out-of-sample performance of the TSK model in estimating different option types.

Notes: The pricing error \((\hat{c}_t - c_t)^2\) across the testing data is shown on the vertical axis. The option types are as follows: (1) Out-of-the-money, short term options, (2) Near-the-money, short term options, (3) In-the-money, short term options, (4) Out-of-the-money, medium term options, (5) Near-the-money, medium term options, (6) In-the-money, medium term options, (7) Out-of-the-money, long term options, (8) Near-the-money, long term options, and (9) In-the-money, long term options.

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