Analog Modeling Complex Dynamical Systems Using Simple Electrical Circuits

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ABSTRACT

Simple analog electrical circuit representing complex dynamical system is described. As case study an array of thirty FitzHugh–Nagumo coupled electronic oscillators, imitating dynamics of neurons, is considered. Due to the parallel processing analog simulation exhibits very fast performance compared with the numerical integration.

Keywords: Analog Computing, Analog Models, Analog Electrical Circuits, Oscillators and Networks.

1. INTRODUCTION

There are many examples in science and engineering where analog electrical circuits have been used to model temporal evolution of dynamical systems. This modeling method has been applied to diverse disciplines and areas. The Mackey–Glass delay differential equation, well known to describe hematological disorders, has a simple electronic analog [1]. A simple electrical circuit has been suggested [2] to imitate the chaotic behavior of a periodically forced mechanical system, described by the Duffing–Holmes equations. Several electrical circuits have been proposed to model the dynamics of neurons [3, 4]. A very interesting solution has been suggested to model mammalian cochlea using high order electrical circuit [5]. One more example is the electrical circuit modeling movement of a spacecraft at the Lagrange point of the astrodynamical Sun–Earth system [6]. It should be emphasized that design of such electrical circuits is not for its own purpose. The analog circuits mentioned above have been employed for testing various methods developed to control dynamics of the systems, specifically to stabilize either steady states [6-9] or periodic orbits [10]. In addition, experiments with electronic analogs can help to better understand the mechanisms behind the behaviors of complex systems, e.g. pitch in human perception of sound [11]. Moreover, the electronic cochlea provides an efficient design of an artificial hearing sensor [12].

One can argue that there is no difference between an analog electrical circuit, imitating a dynamical system, and an analog computer, solving corresponding differential equations. We note that any analog computer is a standard collection of the following main blocks: inverting RC integrators, inverting adders, invertors, inverting and noninverting amplifiers, multipliers, and piecewise linear nonlinear units. Programming of the differential equations on an analog computer is simply wiring these units according to strictly predetermined rules. Differences between the "intrinsically" analog electrical circuits, simulating behavior of dynamical systems, and the conventional analog computers were discussed by Matsumoto, Chua and Komuro 25 years ago in [13]. In this regard it makes sense to present here an excerpt of their paper: “… the circuit ... is not an analog computer in the sense that its building blocks are not integrators. They are ordinary circuit elements; namely, resistors, inductors and capacitors. Both current and voltage of each circuit element play a crucial role in the dynamics of the circuit. On the contrary, the variables in a typical analog computer are merely node voltages of the capacitor–integrator building–block modules, where the circuit current is completely irrelevant in the circuit’s dynamic operation. Hence it would be misleading to confuse our circuit as an analog computer...” [13].
In the present paper, we describe an analog model presented as case study, namely a network of simple electrical circuits, which imitates dynamics of interacting neurons.

2. MATHEMATICAL MODEL

The mathematical model is given by a set of coupled ordinary differential equations [13]:

\[
\begin{align*}
\dot{x}_i &= ax_i - F(x_i) - y_i - b_i + c(<x>-x_i), \\
\dot{y}_i &= x_i - dy_i, & i = 1, 2, \ldots, N, \quad <x> = \sum x_i / N, \\
F(|x_i| \leq 1) &= 0, \quad F(x_i < -1) = d_1 x_i, \quad F(x_i > 1) = d_2 x_i.
\end{align*}
\]

Here \(c\) is the coupling factor; \(N\) is the number of cells.

3. NUMERICAL SIMULATION RESULTS

Numerical simulations for \(N = 30\), \(a = 3.4\), \(b_i = 3 - 0.05(i - 1)\), \(d = 0.15\), \(d_1 = 60\), \(d_2 = 3.4\) were performed using the FREEPASCAL software with the integration step \(h = 10^{-4}\). Typical results are presented in Figs. 1–4.

Fig. 1. Lissajous figure (phase portrait) \([x_i(t) \text{ vs. } x_j(t), i \neq j]\), non-synchronized case, \(c = 0\); 1500 periods of the variable \(x_i(t)\).

Fig. 2. Poincaré section \([x_i(t) \text{ vs. } x_j(t)]\) at \(x_k(t) = 1 \pm 0.01, \frac{dx_k(t)}{dt} < 0, i \neq j, i \neq k, j \neq k\), non-synchronized case, \(c = 0\); 1500 dots.

4. ELECTRICAL CIRCUITS

A network of 30 mean-field coupled (star configuration) electronic neurons is sketched in Fig. 5.

Fig. 3. Lissajous figure (phase portrait) \([x_i(t) \text{ vs. } x_j(t), i \neq j]\), synchronized case, \(c = 0.7\); 1500 periods of the variable \(x_i(t)\).

Fig. 4. Poincaré section \([x_i(t) \text{ vs. } x_j(t)]\) at \(x_k(t) = 1 \pm 0.01, \frac{dx_k(t)}{dt} < 0, i \neq j, i \neq k, j \neq k\), synchronized case, \(c = 0.7\); 1500 dots.

Fig. 5. Block diagram of the mean–field coupled coupled neurons. Coupling resistors \(R^*\) are tuneable units. Node \(O\) is the common coupling node.
Fig. 6 depicts the FitzHugh–Nagumo type electronic circuit representing the individual neurons. The neuron cells in the array are slightly mismatched, either by setting different parameters of the LC tanks or by different external dc biasing of the circuit (corresponds constant term $b_i$ in the differential equations), meeting the fact that in real world there are no strictly identical units.

Fig. 6. Circuit diagram of an electronic analog of a neuron. Element labeled with $-\text{R}$ is a negative resistance, implemented by means of a negative impedance converter.

General view of the hardware circuit is shown in Fig. 7. The construction contains four floors. The electronic neurons are on the ground, the 1st, and the 3rd floor (10 neurons on each floor), while all the coupling resistors $R^*$ are placed on the 2nd floor. Output waveform from a single neuron is presented in Fig. 8.

<table>
<thead>
<tr>
<th>No.</th>
<th>$f_i$, kHz</th>
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<th>$f_i$, kHz</th>
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<th>$f_i$, kHz</th>
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5. ANALOG SIMULATION RESULTS

In the case of weak coupling (large $R^*$) the neurons are spiking independently at their individual frequencies (see Table 1). The intricate Lissajous figure (Fig. 9), multi-dot Poincaré section (Fig. 10), and multiple discrete lines in the spectrum (Fig. 11) confirm this statement.

Fig. 7. Hardware prototype of the array of 30 “neurons”. Dimensions $W \times H \times D = 27 \text{ cm} \times 8 \text{ cm} \times 6.5 \text{ cm}$ (10.6”$\times$3.1”$\times$2.6”)

Fig. 8. Typical output waveform from a single electronic neuron $V_i(t)$. Spike amplitude $\approx 10 \text{ V}$, period $T \approx 80 \mu\text{s}$, frequency $f = 1/T \approx 12.5 \text{ kHz}$.

Fig. 9. Lissajous figure (phase portrait) [$x_i(t)$ vs. $x_j(t), i \neq j$], non-synchronized case. Exposure time $1/8 \text{ s} = 0.125 \text{ s}$. 1500 of snapped periods.

Fig. 10. Poincaré section [$x_i(t)$ vs. $x_j(t)$ at $x_k(t) = 1$, $dx_k(t)/dt < 0$, $i \neq j$, $i \neq k$, $j \neq k$], non-synchronized case. Exposure time $1/8 \text{ s} = 0.125 \text{ s}$. 1500 snapped dots (1 dot per period of variable $x_k(t)$).
Fig. 11. Frequency spectrum $S(f)$ of the mean field $\langle x \rangle$ in the full range from 11.5 kHz to 14.5 kHz, non-synchronized case. 30 lines from the electronic neurons $i = 1, 2, \ldots, 30$. Spectral lines fill the band from $\approx 12.3$ kHz to $\approx 13.8$ kHz. Horizontal scale: linear 500 Hz/div. Spectral resolution $\Delta f = 3$ Hz. Vertical scale: linear.

For stronger coupling (smaller $R^*$) all neurons become fully synchronized, i.e. phase-locked, as is evident from the Lissajous figure displaying a simple ellipse (Fig. 12), a single dot in the Poincaré section (Fig. 13), and the spectrum containing a single line (Fig. 14).

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Fig. 12. Lissajous figure [$x_i(t)$ vs. $x_j(t)$, $i \neq j$]. Synchronized case. Exposure time 1/8 s = 0.125 s. 1500 snapped periods.

Fig. 13. Poincaré section [$x_i(t)$ vs. $x_j(t)$ at $x_k(t) = 1$, $dx_k(t)/dt < 0$, $i \neq j$, $i \neq k$, $j \neq k$], synchronized case. Exposure time 1/8 s = 0.125 s. 1500 snapped dots (1 dot per period of variable $x_k(t)$).

Fig. 14. Frequency spectrum $S(f)$ of the mean field $\langle x \rangle$ in the full range from 11.5 kHz to 14.5 kHz, synchronized case for $i = 1, 2, \ldots, 30$. Single spectral line from the all thirty oscillators at $f_0 \approx 12$ kHz (higher harmonics at $\approx 24$ kHz, $\approx 36$ kHz, ... are out of the shown range). Horizontal scale: linear 500 Hz/div. Spectral resolution $\Delta f = 3$ Hz. Vertical scale: linear.

In addition to the main simulation results presented in Figs. 9–14, we have taken frequency spectrum in a zoomed frequency scale (Fig. 15) to be sure that the spectrum is not continuous one (as observed in noisy and chaotic systems), but has well defined discrete structure.

Fig. 15. Frequency spectrum $S(f)$ of the mean field $\langle x \rangle$, detailed view in the frequency range from 12.60 to 12.90 kHz, non-synchronized case. Seven spectral lines from the oscillators $i = 7, 8, \ldots, 13$. Horizontal scale: linear 50 Hz/div. Spectral resolution $\Delta f = 3$ Hz. Vertical scale: linear.

We note that in order to display analog simulation results presented in Figs. 9-15 from the hardware circuit in Fig. 7 one needs some special electronic equipment. Namely, an oscilloscope with an “X” input (horizontal deflection channel) and a “Y” input (vertical deflection channel) is required to take the Lissajous figures. An “X–Y” channeled oscilloscope with an additional feature of external beam modulation (“Z” input) is necessary (along with an external pulse generator) to plot the Poincaré sections. Finally, either a digital oscilloscope with an integrated Fast Fourier Transform function or an analog spectrum analyzer is needed to get the frequency spectra. However this equipment may not be available in a standard scientific or engineering laboratory.
To get around the problem a standard multi–channel (at least 2–channel) oscilloscope can be used for displaying the waveforms \( x_i(t) \) and \( x_j(t) \) from the pairs of neurons, \( i \neq j \), as shown in Fig. 16 and Fig. 17. The internal horizontal sweep saw–tooth generator of the oscilloscope should be synchronized with one of the input waveform, either to \( x_i(t) \) or to \( x_j(t) \). The waveforms can be inspected visually on the screen of the oscilloscope and photos can be taken, if necessary.

![Fig. 16. Waveforms \( x_i(t) \) (top) and \( x_j(t) \) (bottom), \( i \neq j \), non-synchronized case. Exposure time 1/8 s = 0.125 s. 250 snapped sweeps. Each waveform is 125 sweeps (750 periods of \( x_i(t) \)). Oscilloscope is synchronized internally to \( x_i(t) \).](image)

![Fig. 17. Waveforms \( x_i(t) \) (top) and \( x_j(t) \) (bottom), \( i \neq j \), synchronized case. Exposure time 1/8 s = 0.125 s. 250 snapped sweeps. Each waveform is 125 sweeps (750 periods of \( x_i(t) \)). Oscilloscope is synchronized internally to \( x_i(t) \).](image)

However, this method (also the previously described Lissajous plots and Poincaré sections techniques) requires checking the state of all different pairs of the oscillators \( (i \neq j) \). It maybe time consuming procedure, since there are \( P = N \times (N-1)/2 \) pairs in the network of \( N \) neurons. In the case of \( N = 30 \) there are \( P = 435 \) pairs.

Therefore we propose one more extremely simple alternative technique for checking the network, whether it is in either non-synchronized or synchronized state. One needs a simple single–channel oscilloscope only. Instead of checking the all \( P = 435 \) pairs, the method makes use of a single measurement only. Examples are shown in Fig. 18 and Fig. 19. In the non-synchronized case the mean–field voltage \( <x> \) taken from the node \( O \) has relatively low amplitude \( (<1 \text{ V}) \) and can not synchronized by the oscilloscope (Fig. 18). In contrast, the synchronized mean–field voltage \( <x> \) has relatively high amplitude \( (>10 \text{ V}) \); it is easily synchronized on the screen of the oscilloscope (Fig. 19); exhibits spiking waveform similar to that of a single neuron (see Fig. 8.)

![Fig. 18. Waveform of the mean field \( <x> \), non-synchronized case. Exposure time 1/8 s = 0.125 s. 250 snapped sweeps. Oscilloscope is not able to synchronize to this intricate non-periodic waveform.](image)

![Fig. 19. Waveform of the mean field \( <x> \), synchronized case. Exposure time 1/8 s = 0.125 s. 250 snapped sweeps (1500 periods). Oscilloscope easily synchronizes to this simple periodic waveform.](image)

### 6. COMPARISON BETWEEN NUMERICAL AND ANALOG SIMULATIONS

One of the most important characteristics of any modeling technique is the simulation time required to obtain the result. This parameter is given in Table 2 to compare the two considered methods, numerical versus analog.

<table>
<thead>
<tr>
<th>Plot type</th>
<th>Numerical method</th>
<th>Analog method</th>
</tr>
</thead>
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<tr>
<td>Lissajous figure (1500 periods)</td>
<td>8 min. 20 s</td>
<td>0.125 s</td>
</tr>
<tr>
<td>Poincaré section (1500 dots)</td>
<td>33 min. 40 s</td>
<td>0.125 s</td>
</tr>
<tr>
<td>Waveform ( x(t) ) (1500 periods)</td>
<td>8 min. 20 s</td>
<td>0.125 s</td>
</tr>
</tbody>
</table>

* A standard PC with Intel 1.7 GHz central processor was used.
7. CONCLUDING REMARKS

We have designed, built and investigated an electrical network consisting of $N$ FitzHugh–Nagumo oscillators coupled in a star configuration, and have demonstrated the synchronization effect. The following was taken into account when choosing the number of oscillators $N = 30$. The minimal number of units in a star configuration is $N_{\text{min}} = 3$ [14]. On one hand, number 30 is by an order larger than $N_{\text{min}}$. Thus, 30 oscillators are sufficient to be treated as large array. On the other hand, 30 oscillators is rather small amount of electrical units that can be easily and inexpensively built for a scientific laboratory.

It is evident from Table 2 that the analog simulation technique has great advantage against the numerical simulation from the point of view of time consumption. This is due to fact that analog simulation uses parallel processing; it is independent on $N$, the number of the units. Meanwhile, numerical simulation employs series processing; the processing time is proportional to $N$. Moreover, analog simulations operate with continuous flows on a continuous time scale, in contrast to numerical methods, which require discretization of time and other variables by using small integration step.

In contrast to analog computers, the analog modeling described in the paper is based on some specific analog electrical circuit for a given dynamical system or given differential equation. Despite its limitation to specific systems or equations, electrical circuits have an attractive advantage of simplicity and cheapness. Such circuits comprise rather small number of simple electrical components: resistors, capacitors, inductors, semiconductor diodes, may include operational amplifiers.

The analog modeling method using simple electrical circuits can be applied to many other dynamical systems, especially to complex networks.

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