Uniqueness of Optimal Finite Dimensional Flows for Mixing Enhancement

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ABSTRACT

The problem of mixing enhancement is considered by tuning finite dimensional flows in an optimal way, which is defined as the linear combination of linearly independent steady flows. Optimal mixing control problem is formulated by using the flow as control variable. Using variational principles, we prove the existence of an optimal flow and derive an optimality system that consist of nonlinear convection-diffusion equations and ordinary differential equations. If the initial concentrations are sufficiently small or the diffusivity is sufficiently large, we prove that the optimal flow is unique, and then synthesize an optimal dynamical state feedback controller following the dynamical programming procedure.

Keywords: Mixing Enhancement, Optimal Control, Convection-Diffusion Equations, Variational Principle and Finite-Dimensional Flows.

I. INTRODUCTION

Mixing enhancement is central to the vast majority of processes in the chemical, pharmaceutical, aeronautical, and hydrocarbon processing industries. Well-mixed chemical reactions can yield substantial product benefits and enhanced mixing of fuel can optimize combustion chamber.

Mixing can be enhanced by destabilizing a flow [4], [12], [13], [14], [15], [17], [27], [30]. The flow can be destabilized using passive control devices such as the backward facing step [32] and lobed nozzles [8], open-loop active excitations through flaps and wall-jets [16], and active feedback controls [1], [5], [33], [31].

The objective of this paper is to continue the first author's work [23] by deriving mathematical criteria for an optimal finite dimensional flow for mixing enhancement and proving the uniqueness of the optimal flow under certain conditions, which was left as an open problem in [23].

A simple mathematical mixing model is the convection-diffusion equation

$$\begin{aligned} \frac{\partial c}{\partial t} + (\mathbf{v}\cdot\nabla)c &= \kappa\nabla^2 c \quad \text{in } \Omega, \\ c(\mathbf{x},t_0) &= c^0(\mathbf{x}) \quad \text{in } \Omega, \\ \frac{\partial c}{\partial \mathbf{n}} &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1}$$

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in the absence of a source or sink. In the above equation, $c = c(\mathbf{x},t)$ denotes the concentration of a physical quantity called a passive scalar, $c^0(\mathbf{x})$ is an initial concentration, $\kappa > 0$ denotes the molecular diffusivity of the scalar, Ω is a bounded domain in $\mathbb{R}^n$, $\frac{\partial}{\partial \mathbf{n}}$ denotes the normal derivative along the boundary $\partial\Omega$ ($\mathbf{n}$ denoting the unit normal on the boundary), $\mathbf{v} = \mathbf{v}(\mathbf{x},t)$ denotes an incompressible velocity field ($\nabla\cdot\mathbf{v} = 0$), $\nabla = \left(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_n}\right)$, and $\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$. We assume that $\mathbf{v}$ satisfies no-slip boundary conditions on the boundary $\partial\Omega$ ($\mathbf{v} = 0$).

In the preliminary study [23], we assumed that an arbitrary unsteady flow can be generated. This may not be realistic. Thus we consider finite dimensional flows given by

$$\mathbf{v} = \sum_{i=1}^{m} \bar{\mathbf{v}}_i(\mathbf{x})u_i(t), \tag{2}$$

where $\bar{\mathbf{v}}_i(\mathbf{x})$ $(i = 1,\cdots m)$ are given steady flows and $u_i(t)$ $(i = 1,\cdots m)$ are weight controls. These steady flows prescribe how the control action is distributed in the flow field. Such finite dimensional flows were suggested in the preliminary study [23] and were studied in [24].

Following our preliminary study [23], we define mixing efficiency functionals by penalizing the average of variance of a diffusive scalar and the average of the flow weights. Using variational principles, we prove the existence of an optimal flow weight and derive an optimality system that consists of nonlinear convection-diffusion equations and ordinary differential equations. Furthermore, we show that if the initial concentrations are sufficiently small or the diffusivity is sufficiently large, then the optimal weight is unique. This uniqueness result enables us to synthesize an optimal dynamical state feedback controller following the dynamical programming procedure.

The optimal mixing problem has been studied in the literature. Using the entropy of automorphisms of dynamical systems as the measure of mixing efficiency, D'Alessandro, Dahleh, and Mezic [2] formulated an optimal mixing problem by maximizing the entropy among all permissible periodic sequences composed of two shear flows orthogonal to each other. They derived the form of the protocol which maximizes the entropy by developing appropriate ergodic-theoretic tools. Another optimal mixing problem was defined by Noack, Mezic, Tadmor, and Banaszuk [26], who used the flux across a recirculation region as the measure of mixing efficiency and then maximized the flux among all permissible controlled vortex motions.

The paper is organized as follows. We define a mixing efficiency functional in Section II. We then derive an optimality system for the optimal flow weight in Section III. Using the Banach fixed point theorem of contraction mapping, in Section IV, we prove the uniqueness of the optimal flow weight if initial concentrations are sufficiently small or the diffusivity is sufficiently large.

II. Mixing Efficiency Functionals

We need a number of function spaces for our discussions. We denote by $H^s(\Omega)$ the usual Sobolev space [11] for any $s \in \mathbb{R}$. For $s \geq 0$, $H_0^s(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ in $H^s(\Omega)$, where $C_0^\infty(\Omega)$ denotes the space of all infinitely differentiable functions on Ω with a compact support in Ω. We set

$$\begin{aligned}
\mathbf{L}^2(\Omega) &= \{L^2(\Omega)\}^n, \\
\mathbf{H}_0^1(\Omega) &= \{H_0^1(\Omega)\}^n, \\
\mathbf{H}^2(\Omega) &= \{H^2(\Omega)\}^n, \\
\mathbf{H}_{0,div}^1(\Omega) &= \{\mathbf{v} \in \mathbf{H}_0^1(\Omega) \ : \ \operatorname{div}(\mathbf{v}) = 0 \text{ in } \Omega\}, \\
\mathbf{L}_{div}^2(\Omega) &= \text{ the closure of } \mathbf{H}_{0,div}^1(\Omega) \text{ in } \mathbf{L}^2(\Omega).
\end{aligned}$$

The $\mathbf{L}^2$ norm of a function $\mathbf{f}(\mathbf{x}) \in \mathbf{L}^2(\Omega)$ is denoted by

$$\|\mathbf{f}\| = \left(\int_\Omega |\mathbf{f}(\mathbf{x})|^2 dV\right)^{1/2}.$$

The strain tensor of the velocity $\mathbf{v} = (v_1, v_2, v_3)$ is denoted by

$$\nabla \mathbf{v} = \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{pmatrix}.$$

The mean concentration of $c(\mathbf{x}, t; \mathbf{v})$ is defined by

$$\langle c(t;\mathbf{v}) \rangle = \frac{1}{\operatorname{mes}(\Omega)} \int_\Omega c(\mathbf{x}, t; \mathbf{v}) dV.$$

Let $\bar{\mathbf{v}}_1(\mathbf{x}), \cdots, \bar{\mathbf{v}}_m(\mathbf{x}) \in \mathbf{H}_{0,div}^1(\Omega)$ be a set of linearly independent velocities and $u_1, \cdots, u_m \in H_0^1(0,T)$. As in [23], we define the following mixing efficiency functional

$$\begin{aligned}
& J(u_1, \cdots, u_m) \\
= & \int_0^T \Big(\rho \|c(t;\mathbf{v}) - \langle c(t;\mathbf{v}) \rangle \|^2 + \alpha \|\mathbf{v}(t)\|^2 \\
& + \beta \|\nabla \mathbf{v}(t)\|^2 + \gamma \left\| \frac{\partial \mathbf{v}(t)}{\partial t} \right\|^2 \Big) dt \\
& + \mu \|c(T;\mathbf{v}) - \langle c(T;\mathbf{v}) \rangle \|^2, \qquad (3)
\end{aligned}$$

where $\mathbf{v}$ is given by (2), $T > 0$ is some desired time, and $\alpha > 0, \beta, \gamma, \mu, \rho \geq 0$ are weight constants. For the physical motivation of this functional, we refer to [23].

The weight constants in (3) play an important role in determining the control strength. For small values of α, β, γ, the functional will result in an optimal solution with a small variance of the scalar, but with big magnitudes of the velocity $\mathbf{v}$, of the strain tensor $\nabla \mathbf{v}$, and of the acceleration $\frac{\partial \mathbf{v}}{\partial t}$. This implies that the smaller the weights, the more turbulent the optimal flow, and then the better the mixing enhancement.

There are different measures for mixing efficiency such as Lagrangian and Eulerian time-averages of a flow [3], the mixing variance coefficient [7], and the Mix-Norm defined by Mathew, Mezić, and L. Petzold [25]. For the convenience of treatment of our optimal control problem, we use the L_2 norm of a scalar variance as the mixing efficiency measurement.

We note that the mean is conserved. In fact, integrating equation (1) over Ω gives

$$\frac{d}{dt} \langle c \rangle = \frac{\kappa}{\operatorname{mes}(\Omega)} \int_\Omega \nabla^2 c \, dV = 0,$$

where we have used the boundary conditions on $\mathbf{v}$ and c. Therefore we can assume zero mean without loss of generality. With the zero-mean assumption, the cost functional reduces to

$$\begin{aligned}
& J(u_1, \cdots, u_m) \\
= & \int_0^T \Big(\rho \|c(t;\mathbf{v})\|^2 + \alpha \|\mathbf{v}(t)\|^2 + \beta \|\nabla \mathbf{v}(t)\|^2 \\
& + \gamma \left\| \frac{\partial \mathbf{v}(t)}{\partial t} \right\|^2 \Big) dt + \mu \|c(T;\mathbf{v})\|^2. \qquad (4)
\end{aligned}$$

Then the optimal control problem is to minimize J in an admissible weight space $\mathcal{U} = (H_0^1(0,T))^m$

$$J(u_1^*, \cdots, u_m^*) = \min_{(u_1, \cdots, u_m) \in \mathcal{U}} J(u_1, \cdots, u_m). \qquad (5)$$

The minimizer $(u_1^*, \cdots, u_m^*)$ is called an *optimal weight*.

III. Optimality Systems

The existence of an optimal weight was proved in [23] and the optimality system for the optimal weight can be derived as in [23].

Theorem 3.1: Let $\bar{\mathbf{v}}_1(\mathbf{x}), \cdots, \bar{\mathbf{v}}_m(\mathbf{x}) \in \mathbf{H}_{0,div}^1(\Omega) \cap \mathbf{C}(\Omega)$ be a set of linearly independent velocities. If $(u_1^*, \cdots, u_m^*)$ is an optimal weight under the efficiency functional J defined by (4), then it satisfies the following system

$$\frac{\partial c}{\partial t} + (\mathbf{v}^* \cdot \nabla) c = \kappa \nabla^2 c, \qquad (6)$$

$$\frac{\partial g}{\partial t} + (\mathbf{v}^* \cdot \nabla) g = -\kappa \nabla^2 g + \rho c, \qquad (7)$$

$$\begin{aligned}
& \sum_{i=1}^m \frac{d^2 u_i^*}{dt^2} \int_\Omega \gamma \bar{\mathbf{v}}_j \cdot \bar{\mathbf{v}}_i dV = \int_\Omega g \bar{\mathbf{v}}_j \cdot \nabla c \, dV \\
& + \sum_{i=1}^m u_i^* \int_\Omega \bar{\mathbf{v}}_j \cdot (\alpha \bar{\mathbf{v}}_i - \beta \nabla^2 \bar{\mathbf{v}}_i) dV, \qquad (8)
\end{aligned}$$

$$\mathbf{v}^* = \sum_{i=1}^m \bar{\mathbf{v}}_i(\mathbf{x}) u_i^*(t), \quad j = 1, \cdots, m, \qquad (9)$$

$$\frac{\partial c}{\partial \mathbf{n}} = \frac{\partial g}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega, \qquad (10)$$

$$u_i^*(0) = u_i^*(T) = 0, \quad i = 1, \cdots, m, \qquad (11)$$

$$c(\mathbf{x}, 0) = c^0(\mathbf{x}), \quad g(\mathbf{x}, T) = -\mu c(\mathbf{x}, T). \qquad (12)$$

IV. Uniqueness of Optimal Weight

To prove that the optimal weight is unique, it suffices to prove that the optimality system (6)-(12) has a unique solution (note that the system has at least one solution since an optimal weight exists). We can achieve this by developing a number of estimates about the solutions of (6)-(12).

For convenience, we state a well-known estimate [23] about the solution of (1) as follows.

Lemma 4.1: Let $\mathbf{v} \in L^2(0,T;\mathbf{L}^2_{div}(\Omega))$. Then the solution c of (1) satisfies the following estimate

$$\|c(t)\|^2 + 2\kappa \int_0^t \|\nabla c(s)\|^2 \, ds = \left\|c^0\right\|^2. \tag{13}$$

Lemma 4.2: Let $\bar{\mathbf{v}}_1(\mathbf{x}), \cdots, \bar{\mathbf{v}}_m(\mathbf{x}) \in \mathbf{H}^1_{0,div}(\Omega) \cap \mathbf{C}(\Omega)$ be a set of linearly independent velocities. Let c_1, c_2; g_1, g_2; $(u_1, \cdots, u_m)$, $(w_1, \cdots, w_m)$ be the solutions of (6)-(12) corresponding the initial conditions c_1^0, c_2^0, respectively. Set $\mathbf{u} = \sum_{i=1}^m \bar{\mathbf{v}}_i u_i, \mathbf{w} = \sum_{i=1}^m \bar{\mathbf{v}}_i w_i$. Then the solutions satisfy the following estimates:

$$\begin{aligned}
&\max_{0\le s\le t} \|c_1(s) - c_2(s)\|^2 \\
\le\ & 2\|c_1^0 - c_2^0\|^2 + \frac{2}{\kappa}\|c_2^0\|^2 \int_0^t \|\mathbf{w}(s) - \mathbf{u}(s)\|_\infty^2 ds, \quad (14)\\
&\kappa \int_0^t \|\nabla(c_1 - c_2)(s)\|^2 \, ds \\
\le\ & \|c_1^0 - c_2^0\|^2 + \frac{1}{4\kappa}\|c^0\|^2 \int_0^t \|\mathbf{w}(s) - \mathbf{u}(s)\|_\infty^2 ds, \quad (15)\\
&\max_{0\le s\le t} \|g_1(s) - g_2(s)\|^2 \\
\le\ & \left(\frac{6M^2\rho^2(T-t)}{\kappa^2}\|c_2^0\|^2 + \frac{4\mu^2}{\kappa}\|c_2(T)\|^2\right) \\
&\times \int_0^T \|\mathbf{w}(s) - \mathbf{u}(s)\|_\infty^2 ds + 2\mu^2\|c_1(T) - c_2(T)\|^2 \\
&+ \frac{2M^2\rho^2(T-t)}{\kappa}\|c_1^0 - c_2^0\|^2, \quad (16)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^T \left\| \frac{d}{ds}(u_1(s), \cdots, u_m(s)) \right. \\
&\left. -\frac{d}{ds}(w_1(s), \cdots, w_m(s)) \right\|^2 ds \\
\le\ & \frac{K\|c^0\|^2}{2\kappa} \int_0^T \|g_1(t) - g_2(t)\|^2 dt \\
&+ K\left(\frac{M^2\rho^2 T\|c^0\|^2}{\kappa} + \mu^2\|c_2(T)\|^2\right) \\
&\times \int_0^T \|\nabla c_1(t) - \nabla c_2(t)\|^2 dt, \quad (17)
\end{aligned}$$

where M is the Poincaré's constant in the Poincaré's inequality and K is a positive constant.

Proof. Set $e_c = c_1(\mathbf{u}) - c_2(\mathbf{w})$. A direct calculation shows that e_c satisfies

$$\begin{aligned}
\frac{\partial e_c}{\partial t} + (\mathbf{u}\cdot\nabla)e_c &= \kappa\nabla^2 e_c + ((\mathbf{w}-\mathbf{u})\cdot\nabla)c_2(\mathbf{w}), \quad (18)\\
e_c(\mathbf{x}, 0) &= c_1^0 - c_2^0, \quad \frac{\partial e_c}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega.
\end{aligned}$$

Multiplying (18) by e_c and using the boundary conditions, we obtain the equation

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|e_c(t)\|^2 \\
=\ & -\kappa\|\nabla e_c\|^2 + \int_\Omega e_c((\mathbf{w}-\mathbf{u})\cdot\nabla)c_2(\mathbf{w})dV \\
\le\ & \|\mathbf{w}(\mathbf{t}) - \mathbf{u}(\mathbf{t})\|_\infty \|e_c(t)\| \|\nabla c_2(t;\mathbf{w})\|. \quad (19)
\end{aligned}$$

Integrating over $[0,t]$ gives

$$\begin{aligned}
&\|e_c(t)\|^2 \le \|c_1^0 - c_2^0\|^2 \\
&+2\max_{0\le s\le t}\|e_c(s)\| \\
&\times \int_0^t \|\mathbf{w}(s) - \mathbf{u}(s)\|_\infty \|\nabla c_2(s,\mathbf{w})\| ds \\
\le\ & \|c_1^0 - c_2^0\|^2 \\
&+2\max_{0\le s\le t}\|e_c(s)\| \left(\int_0^t \|\mathbf{w}(s) - \mathbf{u}(s)\|_\infty^2 ds\right)^{1/2} \\
&\times \left(\int_0^t \|\nabla c_2(s,\mathbf{w})\|^2 ds\right)^{1/2},
\end{aligned}$$

which implies that

$$\begin{aligned}
&\max_{0\le s\le t}\|e_c(s)\|^2 \le \|c_1^0 - c_2^0\|^2 \\
&+2\max_{0\le s\le t}\|e_c(s)\| \left(\int_0^t \|\mathbf{w}(s) - \mathbf{u}(s)\|_\infty^2 ds\right)^{1/2} \\
&\times \left(\int_0^t \|\nabla c_2(s,\mathbf{w})\|^2 ds\right)^{1/2} \\
\le\ & \|c_1^0 - c_2^0\|^2 \\
&+\frac{1}{2}\left(\max_{0\le s\le t}\|e_c(s)\|\right)^2 + 2\int_0^t \|\mathbf{w}(s) - \mathbf{u}(s)\|_\infty^2 ds \\
&\times \int_0^t \|\nabla c_2(s,\mathbf{w})\|^2 ds.
\end{aligned}$$

It then follows from (13) that

$$\begin{aligned}
&\max_{0\le s\le t}\|e_c(s)\|^2 \le 2\|c_1^0 - c_2^0\|^2 \quad (20)\\
&+4\int_0^t \|\mathbf{w}(s) - \mathbf{u}(s)\|_\infty^2 ds \int_0^t \|\nabla c(s,\mathbf{w})\|^2 ds \\
\le\ & 2\|c_1^0 - c_2^0\|^2 + \frac{2}{\kappa}\|c_2^0\|^2 \int_0^t \|\mathbf{w}(s) - \mathbf{u}(s)\|_\infty^2 ds.
\end{aligned}$$

This proves (14).

Using the first equation of (19), we derive that

$$\begin{aligned}
&\frac{1}{2}\left\|e_c(t)\right\|^2+\kappa\int_0^t\left\|\nabla e_c(s)\right\|^2 ds\\
\leq\;&\|c_1^0-c_2^0\|^2\\
&+\max_{0\leq s\leq t}\|e_c(s)\|\int_0^t\|\mathbf{w}(s)-\mathbf{u}(s)\|_\infty\|\nabla c_2(s,\mathbf{w})\|ds\\
\leq\;&\|c_1^0-c_2^0\|^2\\
&+\max_{0\leq s\leq t}\|e_c(s)\|\left(\int_0^t\|\mathbf{w}(s)-\mathbf{u}(s)\|_\infty^2 ds\right)^{1/2}\\
&\times\left(\int_0^t\|\nabla c_2(s,\mathbf{w})\|^2 ds\right)^{1/2}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\frac{1}{2}\max_{0\leq s\leq t}\|e_c(t)\|^2+\kappa\int_0^t\left\|\nabla e_c(s)\right\|^2 ds\\
\leq\;&\|c_1^0-c_2^0\|^2\\
&+\max_{0\leq s\leq t}\|e_c(s)\|\left(\int_0^t\|\mathbf{w}(s)-\mathbf{u}(s)\|_\infty^2 ds\right)^{1/2}\\
&\times\left(\int_0^t\|\nabla c_2(s,\mathbf{w})\|^2 ds\right)^{1/2}\\
\leq\;&\|c_1^0-c_2^0\|^2\\
&+\frac{1}{2}\max_{0\leq s\leq t}\|e_c(s)\|^2+\frac{1}{2}\int_0^t\|\mathbf{w}(s)-\mathbf{u}(s)\|_\infty^2 ds\\
&\times\int_0^t\|\nabla c_2(s,\mathbf{w})\|^2 ds.
\end{aligned}$$

It then follows from (13) that

$$\begin{aligned}
&\kappa\int_0^t\left\|\nabla e_c(s)\right\|^2 ds\leq\|c_1^0-c_2^0\|^2\\
&+\frac{1}{4\kappa}\|c^0\|^2\int_0^t\|\mathbf{w}(s)-\mathbf{u}(s)\|_\infty^2 ds.
\end{aligned}\qquad(21)$$

This proves (15).

Set $e_g=g_1(\mathbf{u})-g_2(\mathbf{w})$. A direction calculation shows that

$$\begin{aligned}
&\frac{\partial e_g}{\partial t}+(\mathbf{u}\cdot\nabla)e_g=-\kappa\nabla^2 e_g\\
&+((\mathbf{w}-\mathbf{u})\cdot\nabla)g_2(\mathbf{w})+\rho e_c\quad\text{in }\Omega,\qquad(22)\\
&e_g(\mathbf{x},T)=\mu[c_2(T)-c_1(T)],\ \frac{\partial e_g}{\partial\mathbf{n}}=0\quad\text{on }\partial\Omega.
\end{aligned}$$

Multiplying (22) by e_g and using the boundary conditions, we obtain the equation

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|e_g(t)\right\|^2=\kappa\left\|\nabla e_g\right\|^2\\
&+\int_\Omega e_g((\mathbf{w}-\mathbf{u})\cdot\nabla)g_2(\mathbf{w})dV+\rho\int_\Omega e_g e_c dV\\
\geq\;&\kappa\left\|\nabla e_g\right\|^2-\|\mathbf{w}(\mathbf{t})-\mathbf{u}(\mathbf{t})\|_\infty\|e_g(t)\|\|\nabla g_2(t;\mathbf{w})\|\\
&-\rho\|e_g(t)\|\|e_c(t)\|.
\end{aligned}\qquad(23)$$

Since $\langle e_g\rangle=0$, we have the following Poincarés inequality [10], [11]

$$\|e_g(t)\|\leq M\|\nabla e_g(t)\|.$$

where M is a positive constant. Using the Young's inequality, it therefore follows from (23) that

$$\begin{aligned}
&\frac{d}{dt}\left\|e_g(t)\right\|^2\geq 2\kappa\left\|\nabla e_g\right\|^2\\
&-2\|\mathbf{w}(\mathbf{t})-\mathbf{u}(\mathbf{t})\|_\infty\|e_g(t)\|\|\nabla g_2(t;\mathbf{w})\|\\
&-2M\rho\|\nabla e_g(t)\|\|e_c(t)\|\\
\geq\;&2\kappa\left\|\nabla e_g\right\|^2-2\|\mathbf{w}(\mathbf{t})-\mathbf{u}(\mathbf{t})\|_\infty\|e_g(t)\|\|\nabla g_2(t;\mathbf{w})\|\\
&-2\kappa\|\nabla e_g(t)\|^2+\frac{M^2}{2\kappa}\|e_c(t)\|^2\\
=\;&-2\|\mathbf{w}(\mathbf{t})-\mathbf{u}(\mathbf{t})\|_\infty\|e_g(t)\|\|\nabla g_2(t;\mathbf{w})\|\\
&-\frac{M^2\rho^2}{2\kappa}\|e_c(t)\|^2.
\end{aligned}\qquad(24)$$

Integrating over $[t,T]$ gives

$$\begin{aligned}
&\|e_g(t)\|^2\leq 2\max_{0\leq s\leq t}\|e_g(s)\|\\
&\times\int_t^T\|\mathbf{w}(s)-\mathbf{u}(s)\|_\infty\|\nabla g_2(s,\mathbf{w})\|ds\\
&+\mu^2\|c_1(T)-c_2(T)\|^2+\int_t^T\frac{M^2\rho^2}{2\kappa}\|e_c(s)\|^2 ds\\
\leq\;&2\max_{0\leq s\leq t}\|e_g(s)\|\left(\int_t^T\|\mathbf{w}(s)-\mathbf{u}(s)\|_\infty^2 ds\right)^{1/2}\\
&\times\left(\int_t^T\|\nabla g_2(s,\mathbf{w})\|^2 ds\right)^{1/2}\\
&+\mu^2\|c_1(T)-c_2(T)\|^2+\int_t^T\frac{M^2\rho^2}{2\kappa}\|e_c(s)\|^2 ds.
\end{aligned}$$

It therefore follows from (20) that

$$\begin{aligned}
&\max_{0\leq s\leq t}\|e_g(t)\|^2\leq 2\max_{0\leq s\leq t}\|e_g(s)\|\\
&\times\left(\int_t^T\|\mathbf{w}(s)-\mathbf{u}(s)\|_\infty^2 ds\right)^{1/2}\\
&\times\left(\int_t^T\|\nabla g_2(s,\mathbf{w})\|^2 ds\right)^{1/2}\\
&+\mu^2\|c_1(T)-c_2(T)\|^2+\int_t^T\frac{M^2\rho^2}{2\kappa}\|e_c(s)\|^2 ds\\
\leq\;&\frac{1}{2}\max_{0\leq s\leq t}\|e_g(s)\|^2+2\int_t^T\|\mathbf{w}(s)-\mathbf{u}(s)\|_\infty^2 ds\\
&\times\int_t^T\|\nabla g_2(s,\mathbf{w})\|^2 ds\\
&+\mu^2\|c_1(T)-c_2(T)\|^2+\frac{M^2\rho^2(T-t)}{\kappa}\|c_1^0-c_2^0\|^2\\
&+\frac{M^2\rho^2(T-t)}{\kappa^2}\|c_2^0\|^2\int_0^T\|\mathbf{w}(s)-\mathbf{u}(s)\|_\infty^2 ds,
\end{aligned}\qquad(25)$$

which gives

$$\begin{aligned}
&\max_{0\le s\le t}\|e_g(t)\|^2\\
\le\ & 4\int_t^T \|\mathbf{w}(s)-\mathbf{u}(s)\|_\infty^2 ds\int_t^T \|\nabla g_2(s,\mathbf{w})\|^2 ds\\
&+2\mu^2\|c_1(T)-c_2(T)\|^2+\frac{2M^2\rho^2(T-t)}{\kappa}\|c_1^0-c_2^0\|^2\\
&+\frac{2M^2\rho^2(T-t)}{\kappa^2}\|c_2^0\|^2\int_0^T \|\mathbf{w}(s)-\mathbf{u}(s)\|_\infty^2 ds. \quad (26)
\end{aligned}$$

To estimate $\int_t^T\|\nabla g_2(s,\mathbf{w})\|^2 ds$, we multiply (7) (change g to g_2) by g_2 and integrate over $\Omega\times(t,T)$. Using (13) and the Poincaré's inequality, we can readily derive that

$$\begin{aligned}
&\frac{1}{2}\|g_2(t)\|^2+\frac{\kappa}{2}\int_t^T \|\nabla g_2(s,\mathbf{w})\|^2 ds\\
\le\ & \frac{M^2\rho^2(T-t)}{2\kappa}\|c_2^0\|^2+\frac{\mu^2}{2}\|c_2(T)\|^2. \quad (27)
\end{aligned}$$

It therefore follows from (26) that

$$\begin{aligned}
&\max_{0\le s\le t}\|e_g(t)\|^2\\
\le\ & \left(\frac{6M^2\rho^2(T-t)}{\kappa^2}\|c_2^0\|^2+\frac{4\mu^2}{\kappa}\|c_2(T)\|^2\right)\\
&\times\int_0^T \|\mathbf{w}(s)-\mathbf{u}(s)\|_\infty^2 ds\\
&+2\mu^2\|c_1(T)-c_2(T)\|^2\\
&+\frac{2M^2\rho^2(T-t)}{\kappa}\|c_1^0-c_2^0\|^2. \quad (28)
\end{aligned}$$

This proves (16).

Using the well known estimate [11, Chapter 6] on the boundary value problem (8), it follows from (13) and (27) that there exists a constant K, independent of κ, c^0 such that

$$\begin{aligned}
&\int_0^T \left\|\frac{d}{ds}(u_1(s),\cdots,u_m(s))\right.\\
&\left.-\frac{d}{ds}(w_1(s),\cdots,w_m(s))\right\|^2 ds\\
\le\ & K\max_{0\le t\le T}\|g_1(t)-g_2(t)\|^2\int_0^T\|\nabla c_1(t)\|^2 dt\\
&+K\max_{0\le t\le T}\|g_2(t)\|^2\int_0^T\|\nabla c_1(t)-\nabla c_2(t)\|^2 dt\\
\le\ & \frac{K\|c^0\|^2}{2\kappa}\int_0^T\|g_1(t)-g_2(t)\|^2 dt\\
&+K\left(\frac{M^2\rho^2T\|c^0\|^2}{\kappa}+\mu^2\|c_2(T)\|^2\right)\\
&\times\int_0^T\|\nabla c_1(t)-\nabla c_2(t)\|^2 dt. \quad (29)
\end{aligned}$$

This proves (17).

Theorem 4.1: Let $\bar{\mathbf{v}}_1(\mathbf{x}),\cdots,\bar{\mathbf{v}}_m(\mathbf{x})\in\mathbf{H}^1_{0,div}(\Omega)\cap\mathbf{C}(\Omega)$ be a set of linearly independent velocities. If the initial condition c^0 is sufficiently small or the diffusivity κ is sufficiently large, the optimality system (6)-(12) has a unique solution, and then the optimal weight is unique.

Proof. Suppose the solutions of the optimality system (6)-(12) is not unique. Noting that $c_1^0=c_2^0=c^0$, we deduce from (14), (15), (16), (17), and the Poincaré's inequality that there exists a constant K, independent of κ, c^0 such that

$$\begin{aligned}
&\int_0^T\|(u_1(s),\cdots,u_m(s))-(w_1(s),\cdots,w_m(s))\|^2 ds\\
&\le\frac{K\|c^0\|^4}{4\kappa^3}(12M^2\rho^2T^2+M^2\rho^2T+17\kappa\mu^2)\\
&\times\int_0^T\|(u_1(s),\cdots,u_m(s))-(w_1(s),\cdots,w_m(s))\|^2 ds.
\end{aligned}$$

If the initial condition c^0 is sufficiently small or the diffusivity κ is sufficiently large such that

$$\frac{K\|c^0\|^4}{4\kappa^3}(12M^2\rho^2T^2+M^2\rho^2T+17\kappa\mu^2)<1,$$

then we must have

$$\int_0^T\|(u_1(s),\cdots,u_m(s))-(w_1(s),\cdots,w_m(s))\|^2 ds=0.$$

This is a contradiction.

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