Problems about teaching-learning of differential of a function

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ABSTRACT

A study in Coahuila shows that Bachelors and Masters students do have not much understanding of the concept of differentials and are confused about the concepts of differentials, derivatives and tangent of a function. We have done a detailed study of the didactic obstacles of the concept of differentials, the form in which the concept is treated in classrooms and the difficulties students face in understanding that it is possible to consider the variation of linear functions instead of the variation of the function in certain neighborhood of a point, since this is different for each point. We designed Worksheets for the construction of the concept of differentials with the help of technology, using GeoGebra for functions of one real variable and Matlab for functions of two real variables. They start by working on the visualizations, then questions, theoretical and practical problems, and finally lead to the formalization.

Key words: natural language, mathematical thinking, differential

INTRODUCTION

Many researches have been done around the world on limits, derivatives, and learning them but not much research has been done on differentials of functions and their learning. In this paper we describe teaching-learning problems associated with the concept of differentials. This study is part of a larger project on Calculus learning. It includes Differential Calculus, Integral Calculus and Multivariable Calculus. We have been teaching Calculus since 1969, later in 1989 we began our investigation in mathematical education

RESEARCH QUESTIONS

- Which are the theoretical difficulties found by students that make it quite difficult to correctly understand the concept of differential of a function?

(Cribeiro, 2006) explains the theoretical framework of the Worksheets' design. (Morleth, 2011) designed two Worksheets concerning the concept of differentials.

The motivation is the first stage of Galperin's Theory; we use some context problem known to students. In this moment, students and professor discuss the concept in its intuitive form. Concepts, properties and algorithms, which are the bases for new knowledge, are very important. These things are in second place in Worksheets and using them, students should construct the concept. At this time students should make some advances in recognizing the main characteristics of concepts in different representations associated with a mathematical concept and the process of relating and transforming these representations. The technology's main role is to visualize main characteristics of the concept, after these actions, a synthesizer is necessary.

- Which are the conceptual needs of the students for being able to understand the concepts associated with differential of a function?
- How can the Worksheets be designed in such a way that permits a correct understanding the concept?
- How does the concept of differential of a function contribute to the development of mathematical thinking?

OBJECTIVE

Develop mathematical thinking through learning Calculus

THEORETICAL FRAMEWORK

This study is part of a larger project on Calculus learning based on socio-cultural theory of Vigotski and his evolution of the theory of the stepwise formation of mental actions of Galperin, considered later as a psychological model of the assimilation process to analyze the cognitive activity of teaching. Worksheets were designed using all stages of Galperin's Theory.

> The research carried out by Galperin is of special interest in this paper since his work provided an important step to further investigation and understanding the process of internalization. Galperin's approach shares many similarities with the theoreical framework situated learning, but his approach is, in contrast to most of them, also concerned with how socially and culturally mediated activity is transformed into mental activity (Rambusch Jana, 2006, page 1998).
>
> The idea that learning is essentially a universal process is strongly reinforced by developments in neurobiology. Neurobiologists have demonstrated that changes in behavior can become
>
> permanent because they are stored in the organism's body at a molecular level in the neural system. As a result of acting, special parts of the brain and nervous systems are activated and repeated actions enhance the efficacy of the synapses between the neurons involved in that action. This phenomenon is named long-term potentiation (LTP) (Van Oers, 2008, page 6).

Solving context problems is the way in which students can recognize concepts in order to apply them in science, technology and economy.

All this time, students and professor are working together, but later natural language is the form in which external practical activity can intersect with internal activity. Using natural language, students can explain problems to other students, and these discussions are wonderful and necessary to gain knowledge.

Each student should study individually in order to develop proper mental action. Students will maximize their independence and reach their greatest potential when they do many tasks alone in homework.

METHODOLOGY

I. Observation of Calculus courses during different semesters at two institutions in Coahuila, Mexico in order to establish diagnosis.

During four years we observed students and their primary difficulties in understanding the concept of differential of a function in classes at technological institute "Instituto Tecnológico de Estudios Superiores de la Región Carbonífera". From 2004 to 2008 we observed students at "Universidad Autónoma de Coahuila".

While students are learning the concept of differential of a function in Calculus, main difficulties found are: understanding, expressing and using natural language; and recognizing different representations associated with mathematical concept of the differential of a function in mathematical language.

They need to establish the difference between differential and derivative of a function when the independent variable is near a fixed point. Concepts of neighborhood, intervals, function, domain and range, linear approach, slope of a line, difference between increase of a function and increase of linear function are necessary to understand the concept of differential of a function in order to apply the concept correctly.

It is necessary to speak in natural language with very simple context examples and explain ideas in intuitive form. Students need to recognize the main characteristics of the concept of differential of a function in different representations associated with mathematical concept and the process of relating and transforming these representations.

DIAGNOSIS

Didactic obstacles of Differential

1 - It begins with an unclear definition in terms of contextual and mathematical sense.
2.-The geometrical representation is done in a static board and notebooks, so you only see an arc of curve.
3.-Differential and derivative are confused.
4.-It is not emphasized whether it is a timely, local or global concept.
- Saying differentiable at one point but is taken in a neighborhood for finding the increase of the function.
- When working the geometric representation it is not stressed that the result is not good for any neighborhood.
5.-Different formal definitions do not help clarify the concept when it happens to several variables.
6.-The exercises and operational problems, work the differential by calculating the derivative and multiplying by the differential of the independent variable.
7.-One does not work the idea of linear approximation in a certain neighborhood because it may be invalid for others.
8. - The literature does not express clearly the different key aspects of the concept.

II. Design didactic system

Treatment of concept in classrooms

The definition can be represented in geometrical form for functions of one real variable or functions of real multivariable by a dynamic software like Geogebra or Matlab

Activity Theory vs. Development of capacities and abilities in Worksheets

Worksheets designed using Galperin's Theory of the stepwise formation of mental actions lead to consider the development of capacities and abilities of mathematical thinking in a continuous form. It is a psychological model of the assimilation process to analyze the cognitive activity of teaching.

From the visualization to the formalization

- When is it possible to approximate a function by a linear function?
- At what points can the function be approximated?
- Which properties must meet the function at those points?
- In which neighborhood is this approach valid?
- Under what conditions can I consider that the increase in the linear function is approximately equal to the increase in original function?
- How to link the same idea in different spaces?

Both Worksheets have the same structure. This structure is based on stages of Galperin's Theory: Motivation, orientation phase, expressed in practical activities, dialogical actions, dialogical thinking, and mental action stage. For the practical activities, students work with material support, they and the teacher work together using the concept of zone of proximal development (ZPD), but during dialogical action they have more independence. Dialogical thinking and mental action stage need some individual tasks.

Motivation. Simple context situations can motivate students because they show the concept used in its intuitive form. For example, several problems about approximaton are presented.

Basic concepts. Learning Theories have established that things in cognitive structure are the most important in learning process. Maybe, students do not have the necessary basic concepts in their cognitive structure. Worksheets have tasks to train basic concept, such as, for an undefined function, point x is near to a point p, distance between x and p, neighborhood of point p in function domain, neighborhood of point L in function range.

Construct concept of differential. Students and teacher must discuss the intuitive form of the concept in natural language. Students should take some actions to recognize different representations associated with a mathematical concept (graphic representation, representation with tables, algebraically representation, and formal representation) and the process of relating and transforming these representations. The main role of technology is visualizing the main characteristics of the concept. We use programs in GeoGebra to manipulate points x near to p, and their images y near to L. Students observe the position of y when they move x, and answer questions in worksheets.

Context problems. Solving context problems is the way in which student can recognize how to apply the concept in science, technology and economy.

Synthetize principle ideas. At any given time, students' minds are filling with all sorts of knowledge and it is necessary to establishes ideas in simple form, in order to help students grasp the concepts easier.

Actions in the classroom. Two activities have been designed in the Worksheets. Half of the group works on each one, later they explain the activity to the other group. Dialogical actions are developed in this section of the Worksheet.

Independent work. Tasks in this section are meant to encourage learners to establish dialogue with them and write down their thoughts.

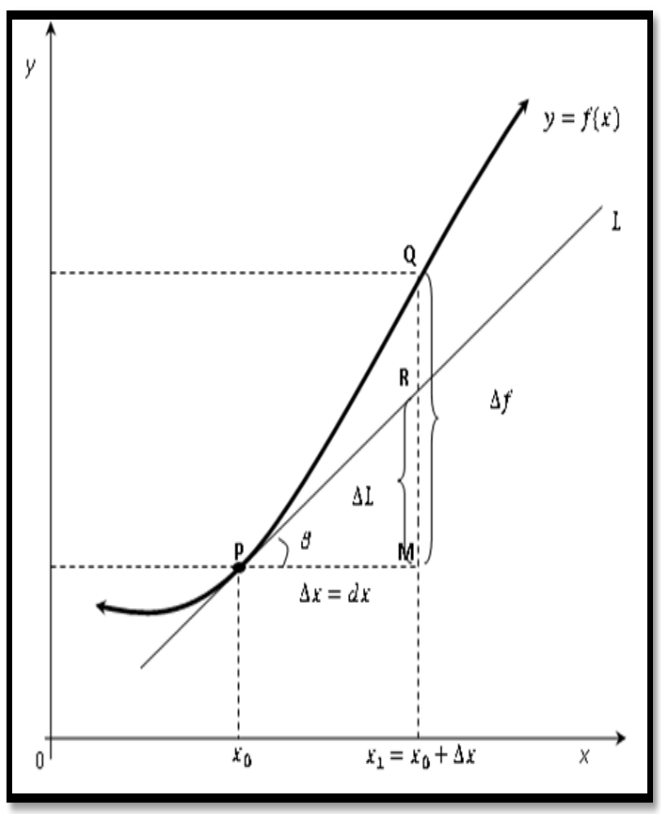
Conclusions. Their thoughts are discussed in an assembly to make a final conclusion.

Phases	Worksheets	Capacities and abilities developed
Motivation	Motivation	Visualization Realistic context and mathematics links
Orientation. Learners get to know the task and its conditions (materialized level) Provide all information necessary for successful execution of the action	• Working with basic concepts • Construction of the concept • Context Problems • Synthetize Ideas • To conjecture	Conduct literature search select material observe identify properties interpret classify compare relate analyze argue integrate knowledge demonstrations to model problem solving synthesize ideas set conjectures
Communicated thinking	Teamwork, discussions	communication expose discussion teamwork
Dialogical thinking with him/herself. A transformation of the structure of speech takes place.	Independent work out of classroom	abstract generalize
Thought activities	Conclusions	reflect conclude

IDEAS TO BUILD THE CONCEPT OF DIFFERENTIAL

- Linear Approximation in certain neighborhoods
- General concepts emphasizing the difference of them according to the dimension of the space
- Conceptual differences between neighborhoods, limit, continuity, derivability and differentiability according to the dimension of the space in which we are working
- Analysis of necessary and sufficient conditions for enforcing a property in different spaces

Formalization of the concept of the differential of a function of one real variable without technology and with Worksheets



Answer the following questions regarding the above illustration

- Express the increment of a function in the interval $[x_0, xo + h] = [x_0, xo + \Delta x]$
- Explain with your own words what the increment of x, the increment of L (ΔL) and the increment of f (Δf) represent.

Build the increment of L (ΔL)

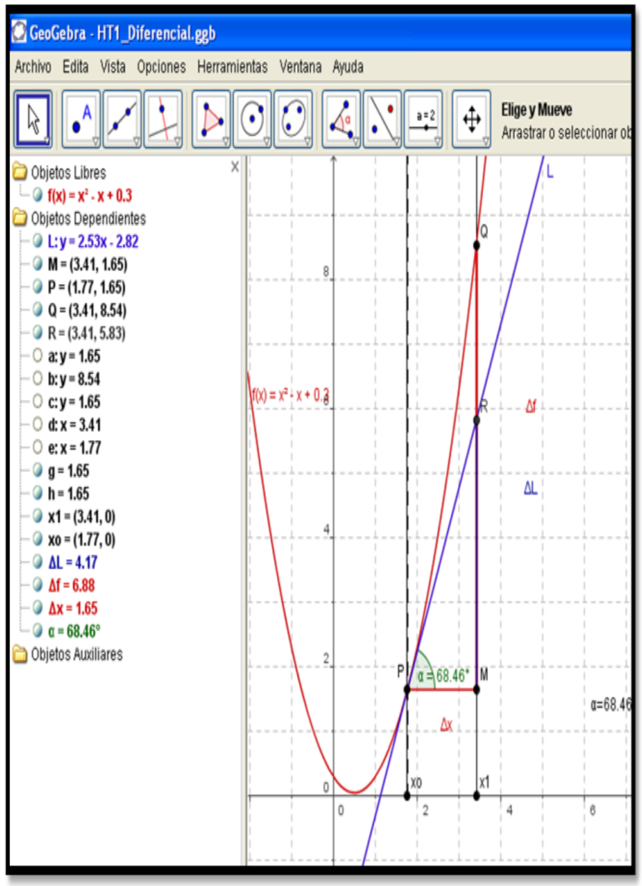
Build an algebraic expression for the sides of the triangle formed by the points P, R and M.
• From the expression of ΔL obtained earlier, what is the value that represents the slope of the line L?
• In the algebraic expression which you obtained, replace the values of RM and PM respectively
• If you know the value of dx = x is infinitesimal. What would be the expression that determines the value of ΔL?

Build the increment of f (Δf)

Now look at the Δf value represented in the illustration. Construct an algebraic expression to determine the value Δf

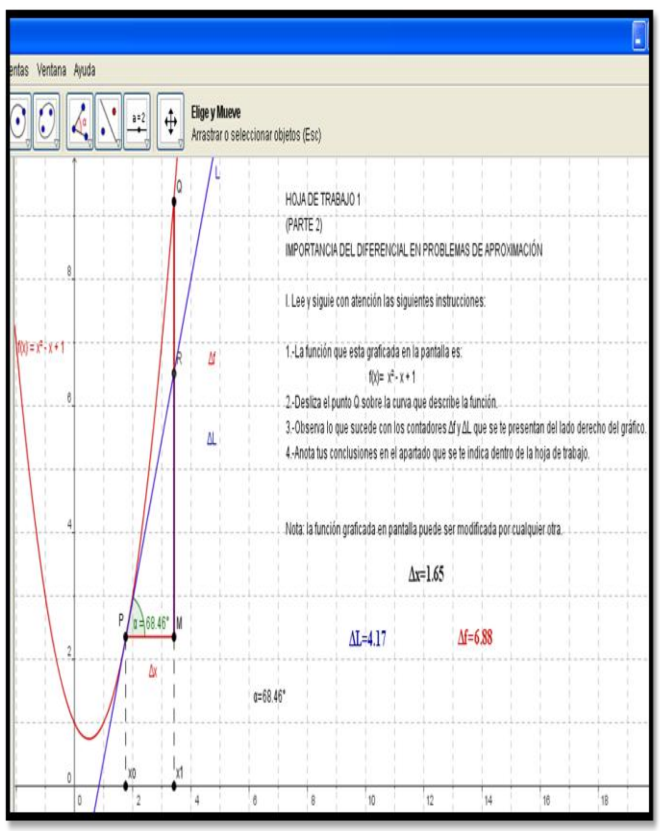
Conclusion on the similarity between the values of Δf y ΔL **when** $\Delta x \longrightarrow 0$

Once you have an algebraic expression to represent the values of Δf and ΔL and you understand the meaning they have in the graph shown in the illustration, answer the following questions:
What happens to the value of ΔL as $\Delta x \longrightarrow 0$ (gets smaller)?
What happens to the value of Δf as $\Delta x \longrightarrow 0$?
What happens between the line graph L and the curve that describes the function f as
$x_1 \rightarrow x_0$?



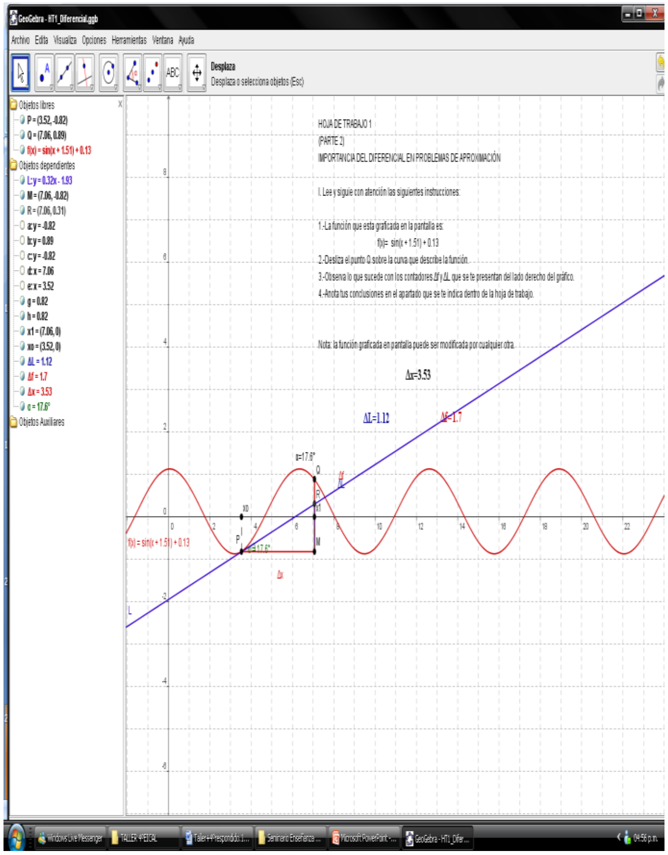
Formalization of the concept of the differential of a function of one real variable with technology and with Worksheets

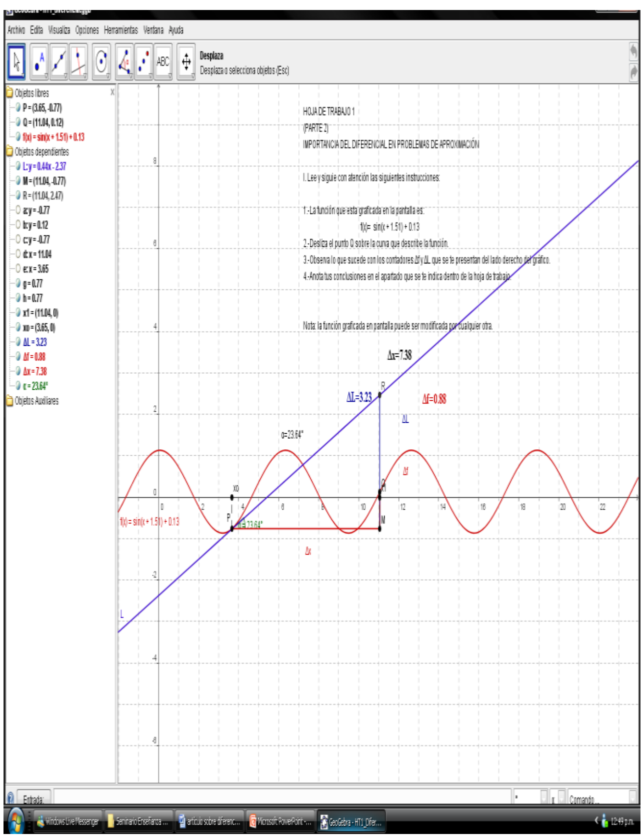
Using the Geometry GeoGebra program called HT1_DIiferencial.ggm compare the value of Δf and ΔL with Δx for different points of the following functions



$y = x^2$

$y = 2sin(x)$



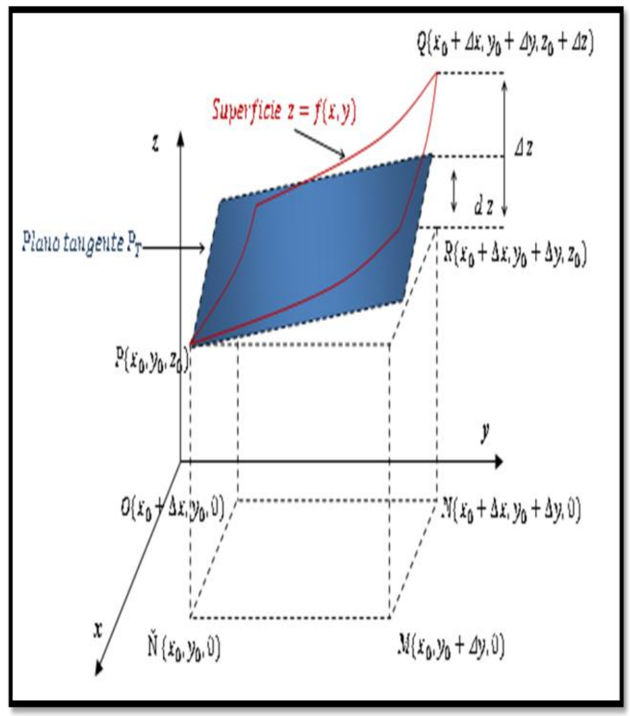


As $\Delta x \longrightarrow 0$ the increments of the tangent line and the increment of the function are very similar. Answer the following question according to your personal analysis with the effective use of the program with the two functions and what you observed about $\Delta x \longrightarrow 0$:

The values of Δx for which can be seen that increases in L and f are almost the same, is due to:

i) An arbitrary personal decision
ii) Growth characteristics of the function
iii) Characteristics of the problem under consideration
iv) All of the above
v) Other reason that you believe

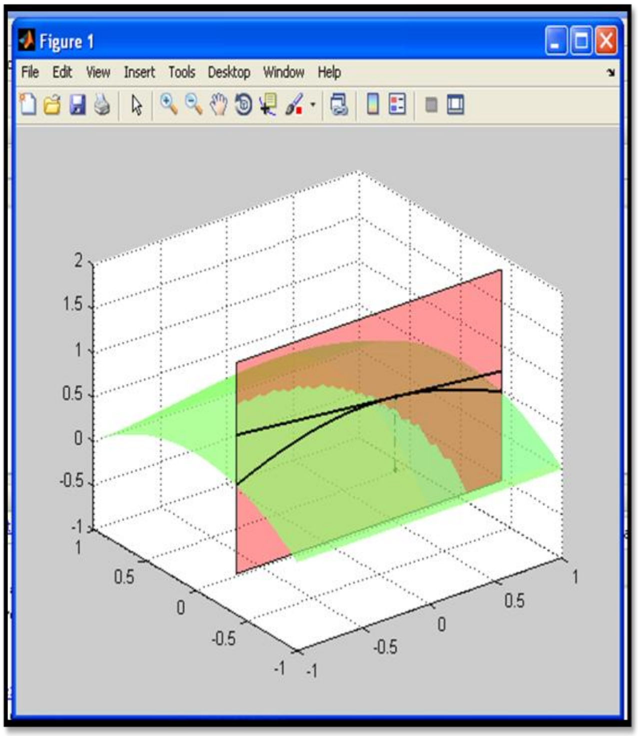
Formalization of the concept of the differential of a function of n real variables without technology and with Worksheets

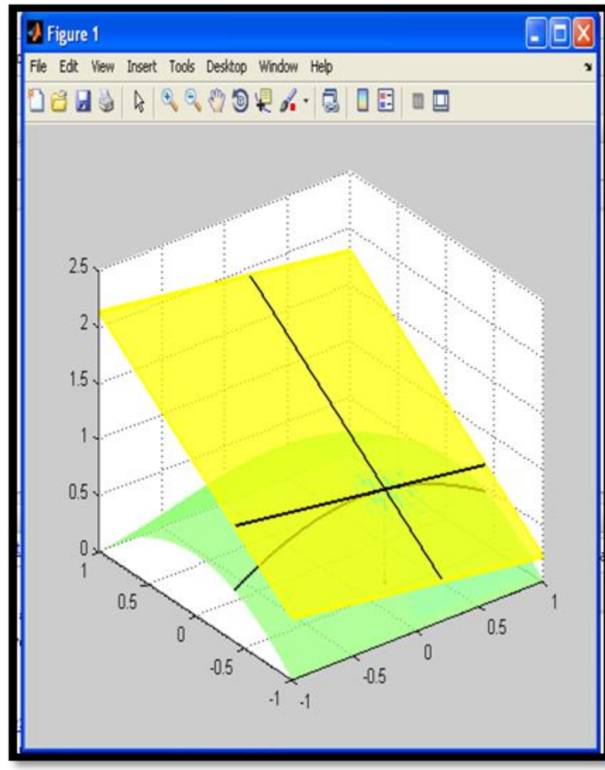


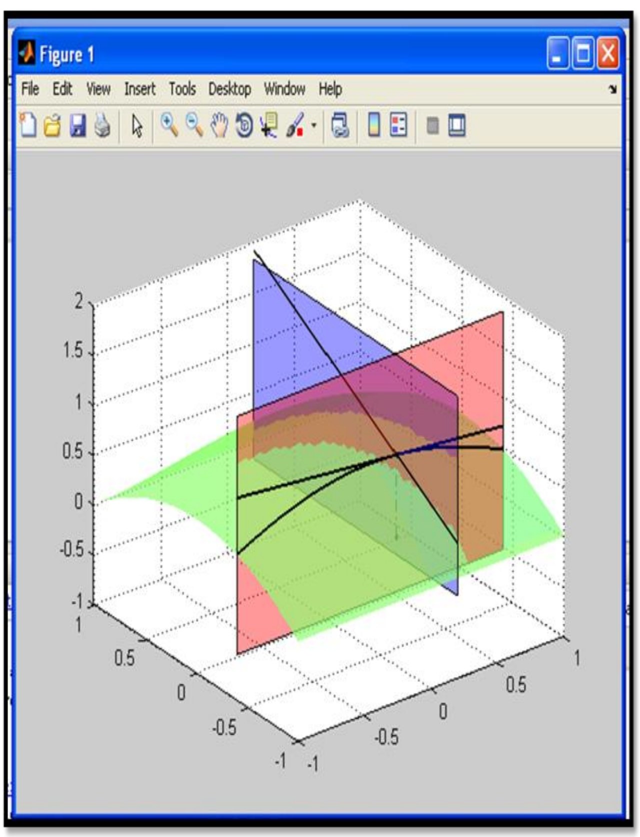
Formalization of the concept of differential of a function of n real variables with technology and with Worksheets

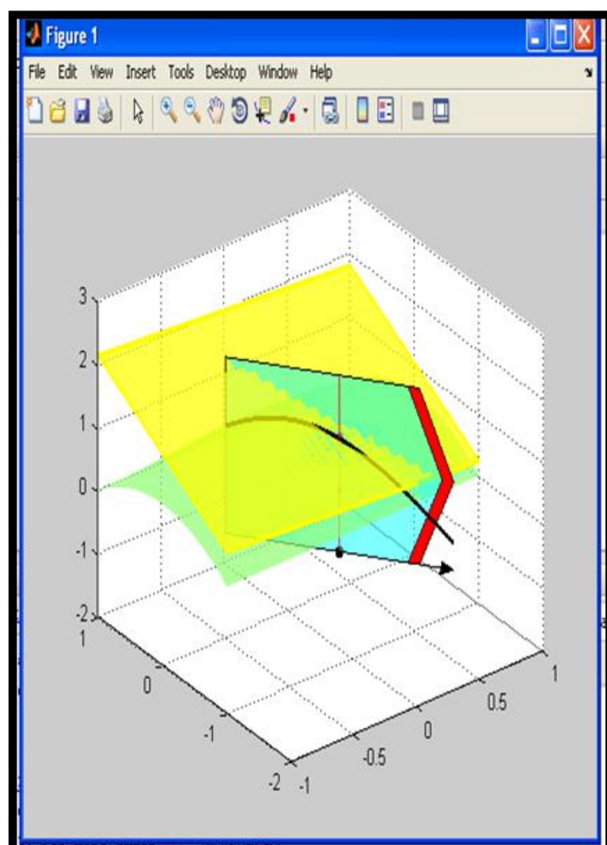
- From the expression for ΔP_T obtained, what is the value that represents the slope of the tangent plane to the surface of the function?
- Establish a parallel line for the case of the differential of a variable such that it represents the slope of the line L.
- What can you conclude from this?
- If you know the value of $dx = \Delta x$ is infinitesimal in the case of the differential of a variable. What would be the expression that determines the value of ΔP_T? Assuming that $dx = \Delta x$ and $dy = \Delta y$ are infinitesimal.

The value of ΔP_T is equal to dz and both represent the differential of the dependent variable of a function.









Using the MatLab program called splano.m compare the value ΔP_T with respect to Δf for different points of the following functions:

i) $z = x^2 y + 2xy^2$

ii) $z = \dfrac{\sin 2x}{1+\cos y}$

As $\Delta x \to 0$, $\Delta y \to 0$, the increments of the tangent plane and of the function become very similar. Answer the following question according to your personal analysis with the effective use of the program with the two functions and what you observed about $\Delta x \to 0$, $\Delta y \to 0$:

The values of y for which the increments ΔP_T and Δf are almost the same, is due to:

- An arbitrary personal decision
- Growth characteristics of the function
- The characteristics of the problem under consideration
- All of the above
- Other reason you believe

THEORETICAL ISSUES

Consider whether the properties that hold true for the case of functions of one variable are true for functions of two variables, taking into account the variation of the concepts of continuity, derivative and differential.

What properties that are true for the case of functions of one variable are also true for functions of two variables? Justify your answer taking into account the variation of the concepts of continuity, derivative and differential.

- Is the continuity of the function necessary for the existence of the differential?
- Is continuity of the function sufficient to ensure that it is differentiable?
- Does the fact that a function is derivable mean that it is also differentiable? Justify your answer and compare it with the response for the case of functions of one variable.
- Is the reciprocal of the previous question true? Answer yes or no, and why.

Making use of theorems and propositions discuss the following properties for one and several variables

$$diferentiable \Rightarrow derivable$$

$$diferentiable \Leftarrow derivable$$

$$diferentiability \Rightarrow continuity$$

$$diferenciability \underset{not}{\Longleftarrow} continuity$$

$$derivable \Rightarrow continuity$$

$$derivable \overset{not}{\Longleftarrow} continuity$$

$$\boldsymbol{diferenciable \leftrightarrow derivable}$$ for functions of one variable

$$diferenciability \underset{not}{\Longleftarrow} continuity$$

Consider the same questions for functions of several variables

Case I: Examples of functions that do not comply with any property that allows differentiability and derivability.

Case II: Examples of functions that meet more than one property required to be differentiable and / or derivable.

Case III: Examples of functions that meet more than one property required to be differentiable and / or derivable

Case IV: Examples of functions of several variables that meet all necessary properties required to be differentiable and / or derivable

III. Apply didactic system and observe results

Didactic system was applied to three groups at Coahuila's University. During the courses we observed change in 75% of students. They show better understanding, improved writing and verbal communication and are able to apply the concepts in a better way.

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