NUMERICAL SIMULATION OF RADIATIVE COOLING IN A 3-D ANISOTROPIC SOLID, USING A SYSTEM OF TWO PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS.

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Abstract:
We consider heat transfer, through radiative cooling, in a 3-D anisotropic solid, without convection. Since thermal conduction is not the same in all directions in an anisotropic solid, the mathematical model based on the cooling model in [1] assumes a vectorial form. In this paper, we show the development of the model and subsequently discretize it using forward finite differences. The numerical scheme developed is tested for convergence and error stability.

Our mathematical modeling is based on the energy conservation laws. In the process we also apply the Fourier and radiation laws. However, due to the results obtained in [2], the Stefan-Boltzmann radiation law will not be utilized. For surface radiation, we apply differential geometry to develop a dynamic boundary equation that will simulate surface radiation itself. The two parabolic PDEs developed are linked through the identity,

\[ \nabla u := \nabla_s \left[ \gamma_0 u + \gamma_1 u n \right], \]

where \( \gamma_0 u \) is the surface temperature and \( \gamma_1 u = \frac{\partial u}{\partial n} \). It is assumed the solid does not receive any external heat energy.

In this paper, we develop and discretize the following model:

(a) \( \partial_t [u] = [B][\Delta u] - \beta \left( (u - U)^\alpha \right) \)

Subject to:

(i) \( \gamma_1 u = 0; \)

(ii) \( u(x,0) = [u^0(x)], \) and,

(iii) \( \partial_t [\gamma_0 u] = [S][\Delta_s [\gamma_0 u]] - \sigma \left( [\gamma_0 u - U]^\alpha \right) \)

where,

(c) \( [B] = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \) a body conductivity matrix; \( [S] = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \) a surface radiative matrix; with \( \beta : \) Hlomuka constant(see[1]); \( \sigma : \) Stefan-Boltzmann constant.
As can be seen in (b)(iii), the model has a dynamic boundary condition which is itself a nonlinear parabolic partial differential equation. As we indicated earlier in this abstract, the system is not autonomous.

Note that the condition (b)(i) could mean the complete insulation of the solid; hence the introduction of the term, \(-\beta[(u - U)\alpha]\), \(\beta > 0\), in the body conduction equation to effect the flow of heat energy from the solid. This method of creating a heat energy flow from an insulated solid was initially proposed, theoretically, in the model in (1). Some scholars were doubtful until it was successfully applied, computationally, in [3] and [4], resulting in a numerical decrease in the body temperature of a solid. We were able to print out (simultaneously) the body and surface temperatures.

As we have indicated in the previous paragraph, prior to this paper, radiative heat transfer was considered in [3] and [4], where schemes capable of printing both body and surface temperatures were developed for isotropic solids.

In this paper we include a numerical example that will simulate the radiation phenomenon itself; print out both body and surface temperatures, as in [3] and [4].

**Symbols used:**

\(\Omega\): open bounded domain in \(\mathbb{R}^3\) with boundary \(\partial \Omega\);

\(x = (x_1, x_2, x_3) \in \Omega; \ s = (s_1, s_2) \in \partial \Omega;\)

\(H^2(\Omega)\): Sobolev space with boundary \(H^2(\partial \Omega)\);

\(u(x, t)\): solid body temperature in \(H^2(\Omega)\);

\(\gamma_s u(s, t)\): solid surface temperature, \(s \in \partial \Omega\);

\(\gamma_t u = \frac{\partial u}{\partial n}; n\) a unit external normal vector to \(\partial \Omega\);
\[ \nabla_s = \frac{\partial}{\partial s_1} \tau_1 + \frac{\partial}{\partial s_2} \tau_2, \ \text{tangential to} \ \partial \Omega \ \text{at an arbitrary surface point} \ (\tau_1, \tau_2, \mathbf{n}); \]

where \( \tau_1, \tau_2 \) and \( \mathbf{n} \) are the coordinates on the tangent space.

\[ \Delta_s = \nabla_s \cdot \nabla_s; \ \text{Beltrami/Laplace surface operator.} \]

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[1]: Joe Hlomuka: On the existence, uniqueness and the stability of a solution to a cooling problem, for an isotropic 3-D solid. 

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[4]. Joe Hlomuka: The finite element algorithm for the nonlinear radiative cooling of a 2-D isotropic solid. 
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