New constants in the system of Verhulst equations

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ABSTRACT

The given work is the continuation of research results presented by the authors at the international World Congress on Science, Engineering and Technology, July 29-31, 2009, Oslo, Norway (WCSET 2009). Initially authors analyzed the dynamics of two antagonistic populations (two different species of fish: pike and perch) with the help of the system of two recurrent equations on the base of Verhulst-Pearl model. During the research the following areas of change of the parameters, guaranteeing realization of some evolutionary situation, were determined: zones of the steady decisions, zone of bifurcation of occurrence and cycles, zone of chaos and uncertainty. During the continuation of the research authors discovered that the similar phenomenon could be found in the activity of financial markets. In this paper the discovery of the new constants in the system of Verhulst equations is presented.

Keywords – bifurcation, chaos, dynamics of populations, fractals.

1. INTRODUCTION

The base discrete model describing limited growth of number of population is the Verhulst model $x_{n+1} = \alpha x_n(1-x_n)$. This model has become the starting point for the whole cycle of works [1,2,3]. Recently [1] the similar phenomena in economy and in financial activity of the markets were analyzed.

Let x_n be the number of the population of one kind and y_n be the number of population of another kind in the year n. Lets now look at the system, containing two iterative equations

$$\begin{cases} y_{n+1} = \alpha\, x_n(1-x_n) \\ x_{n+1} = \beta\, y_{n+1}(1-y_{n+1}) \end{cases}. \tag{1}$$

The detailed explanation of mathematical calculations for this paper can be found in [9]. In the paper [4] the questions of self-destruction within one population type has been discussed. The authors of this article consider the classical works of S.P. Kuznezov [5, 6] to be the closest to their theme of study. The theoretical bases for the self-destruction phenomenon can be found in the classical works [7, 8]).

2. AREAS OF VARIOUS BEHAVIORS

Our research is significantly facilitated if we note that difference system (1) in its essence is the simple iterative scheme for the roots of the system of the nonlinear equations

$$\begin{cases} y = \alpha\, x(1-x) \\ x = \beta\, y(1-y) \end{cases}. \tag{2}$$

The solution of which has the following form

$$x_1 = 0, \qquad x_2 = \frac{2}{3} + \frac{A}{6\beta\alpha} + \frac{2(\alpha-3)\beta}{3A},$$

$$x_3 = \frac{2}{3} + \frac{A(-1+i\sqrt{3})}{12\beta\alpha} - \frac{(\alpha-3)\beta(1+i\sqrt{3})}{3A},$$

$$x_4 = \frac{2}{3} - \frac{A(1+i\sqrt{3})}{12\beta\alpha} + \frac{(\alpha-3)\beta(-1+i\sqrt{3})}{3A},$$

where

$$A = \sqrt[3]{(36\alpha\beta - 8\alpha^2\beta - 108 + 12\sqrt{g})\alpha\beta^2},$$

$$g(\alpha,\beta) = 81 - 54\alpha\beta + 12\alpha\beta^2 - 3\alpha^2\beta^2 + 12\alpha^2\beta. \tag{3}$$

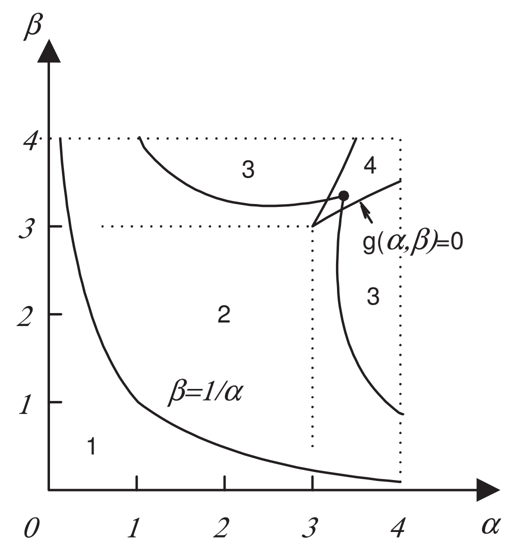

Fig. 1. Demonstrates the dynamic regime of system (1): 1- area, where both kinds of species degenerate; 2- area, where the number of both species is stabilizing; 3- area with the border $g(\alpha,\beta)=0$, where the cycles S=2^1 occur; 4 - areas of the transfer between the cycle areas and the areas of development of the dynamic chaos.

3. ZONES OF CYCLES AND SPECIAL POINTS OCCURRENCE

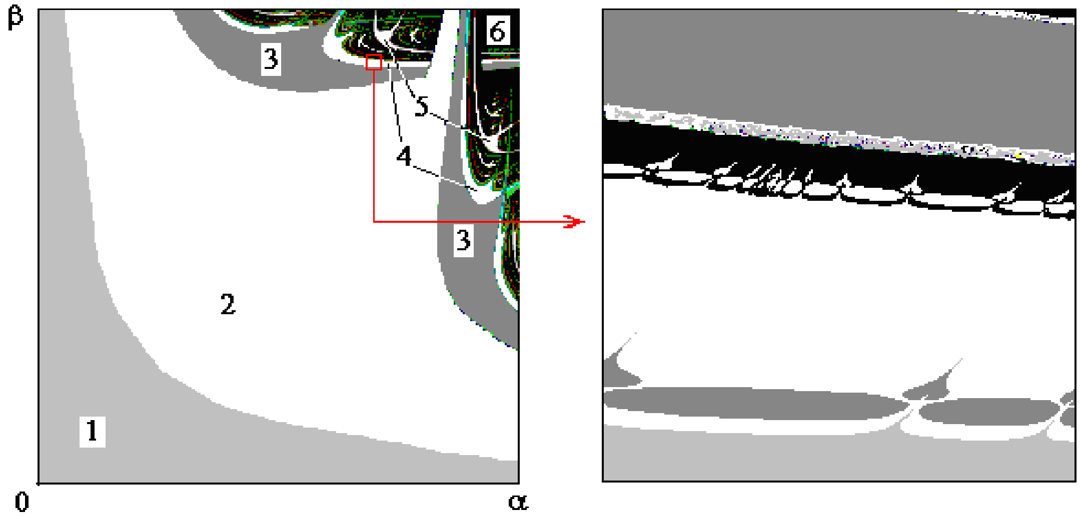

Fig. 2. On the left: the map of the dynamic regime of the system (1). On the right : analogues of Julia Fractals on the border of the zones with several cycle numbers:

1- area, where both kinds of species degenerate;
2- area, where the number of both species is stabilizing;
3- area, where the cycles S=2^1 occur;
4- area, where the cycles S=2^2 and more occur;
5- area, where the cycles S=3^1 and more occur;
6 - areas of the transfer between the cycle areas and the areas of development of the dynamic chaos.

The border of cyclic zones with the different order (Fig.2. Left) gives us in the cross-section (Fig. 2 Right) well known points of branching of series of cycles, where $x_{n+1} = a x_n (1 - x_n)$ for $n \to 2n$: if $a = 3$, then $1 \to 2$; $a = 1 + \sqrt{6}$, then $2 \to 4$; $a = 3{,}543$, then $4 \to 8$; $a = 3{,}563$, then $8 \to 16$; $a = 3{,}568$, then $16 \to 32$.

A series of cycles 2^n comes to an end at $a \approx 3{,}575$. For the large values a the cyclic decisions either are absent, or have very large length.

At $a = 1 + \sqrt{8}$ there are two cycles of length 3 - steady and unstable. At the further growth α length of a steady cycle runs consistently: $3, 6, 12, 24, ..., 3 \times 2^n$. It is also possible to receive the other values of parameter for occurrence of cycles of other length.

During the study of the system equations (1) the fact of occurrence of essentially new points, new constants was discovered by authors:

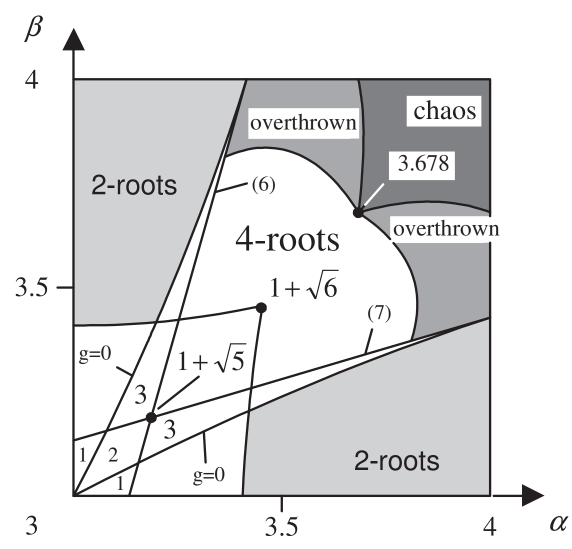

Fig. 3. The dashed line designates the greatest possible width of attractors. The most complex phenomenon occurs at such values of parameters α and β, at which the width of attractors is increasing to the point, where attractors begin to overlap. In this case iterations begin to be overthrown from the ***area of activity*** of one attractor into the ***activity area*** of another attractor.

Equating $x_2 = x_3$, we shall receive the equation of a curve of the beginning of overlapping of attractor. Using symmetry it is possible to accept $x_3 = 1 - x_1$ and to simplify expression: $x_2 = 1 - x_1$ or

$$\frac{\alpha\beta^2}{4}\left(1-\frac{\beta}{4}\right)\left[1-\frac{\alpha\beta}{4}\left(1-\frac{\beta}{4}\right)\right]=1-\frac{\alpha\beta}{4}\left(1-\frac{\alpha}{4}\right)$$

At $\alpha = \beta$ this expression accepts a kind $\frac{\alpha^3}{4}\left(1-\frac{\alpha}{4}\right)-1=0$, with material meaning of the root

$$\alpha = \frac{2}{3}\left(\sqrt{19+3\sqrt{33}}+\frac{4}{\sqrt{19+3\sqrt{33}}}+1\right), \quad \alpha \approx 3{,}678.$$

It is an essentially new point, which does not exist in one-dimensional case. It is a point of random moving of solutions from one zone into another (Fig.4). Formulas of the borderline (6), (7) look like: $\alpha\beta(4-\alpha)=8$, $\alpha\beta(4-\beta)=8$. This is the border of the zone 4 solutions. If $\alpha=\beta$ we have $\alpha^2(4-\alpha)=8$ and $\alpha=\beta=1+\sqrt{5}$. The formula for the borderlines of chaos zone (**Fig.4**), is unknown to authors and at the moment can be only computer approximated.

Consequently, when $\alpha,\beta \geq 3.678$ an additional way of dynamic chaos formation is appeared. The cause of it is overlapping of the attractors, during which the width of the attractors increases up the point, where a coherent interaction of attractors (overthrown from the ***area of activity*** of one attractor into the ***activity area*** of another attractor) can practically have any kind of values (Fig.4).

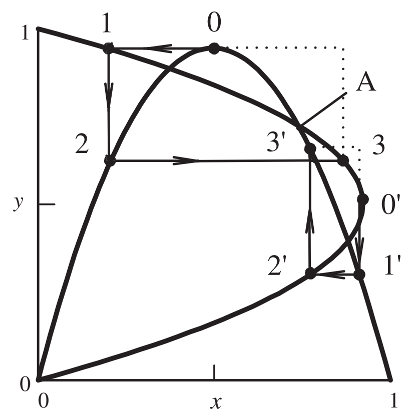

Fig.4. The initial evolution of one cycle proceeds through the conditions: $0 \to 1 \to 2 \to 3$, evolution of another cycle proceeds through conditions: $0' \to 1' \to 2' \to 3'$. Here the key role is played by the point A. It is a point crossing of the diagrams of functions. If the width of the attractor exceeds the excess of the point o (or o') above the point A, then the iterative cycle $0 \to 1 \to 2 \to 3$ cannot reach its own attractor (zone of an attraction around the left root), and will move to operative range of a symmetric root.

However, lets note that even in this case there is a possibility to have a coherent interaction of attractors, which results in existence of simple limited cycles $S = 3^1$ at $\alpha=\beta=1+\sqrt{8}$, or $S = 5^1$ at $\alpha=\beta=3{,}7389$. On the bifurcation diagrams (Fig. 5, 6) such cycles correspond to ***Sharkovsky Windows***. Obviously the given configurations also allow their own cascade of bifurcations, which corresponds to the general theory of differentiable mapping.
The results of our computer modeling are shown in a **Fig. 5.**

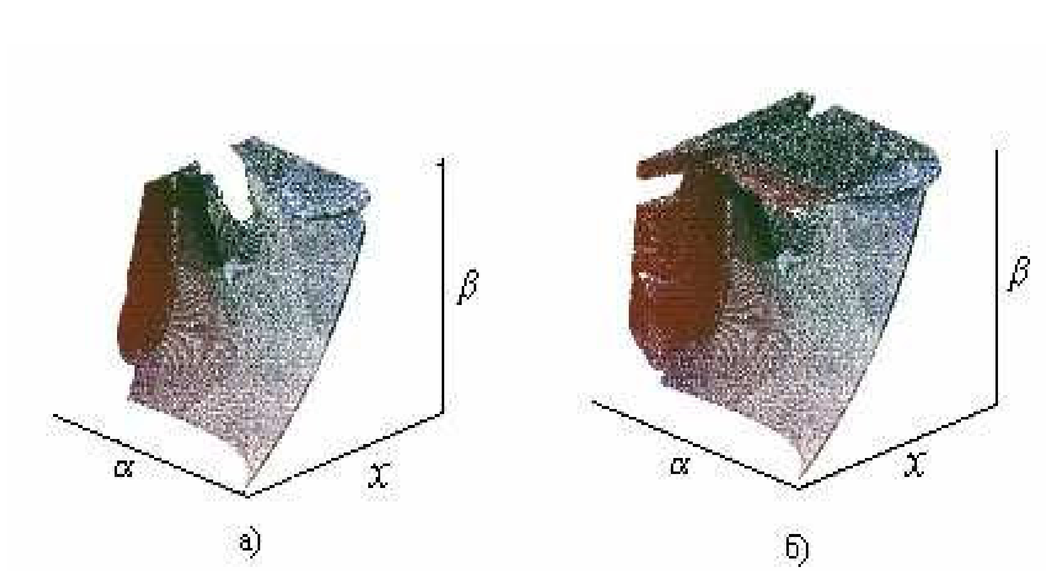

Fig.5. Evolution of the bifurcation diagram of two-dimensional Verhulst mapping a) $\alpha,\beta \leq 3.6$, б) $\alpha,\beta \leq 3.9$. "Fractal cabbage".

The increase of the certain parts of Fig.5. allows to detect the already known phenomena: layered attractors with the fractional Hausdorff dimension, the infinite number of the decreasing in size "windows" of the periodical regimes, forming in cross-sections the iterations of analogues of Sierpinski carpet. Diagonal cross-section at $\alpha = \beta$, as it can be seen on the edge of **Fig.5**., contains the classical diagram of the Feigenbaum fractal (**Fig.6**)

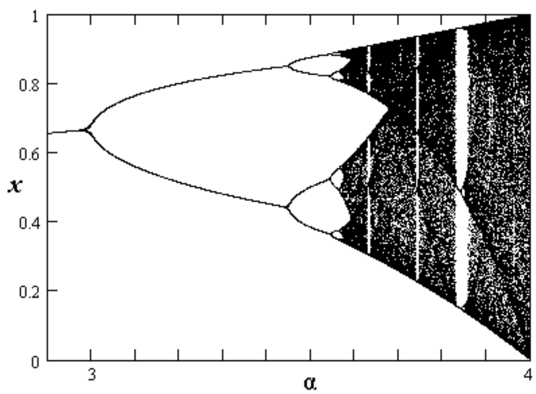

Fig.6. The Feigenbaum tree, $\alpha = \beta$.

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