On the Complexity of Bivariate Lattice with Stochastic Interest Rate Models

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Abstract

Many complex financial instruments with multiple state variables have no analytical formulas and thus must be priced by numerical methods like lattice. The bivariate lattice is a numerical method that is widely used to work with correlated state variables. Some research has focused on bivariate lattices for models with two state variables: stochastic underlying asset prices (e.g., stock prices) and stochastic interest rates. However, when the interest rate model allows rates to grow superpolynomially, the said lattices generate invalid transition probabilities. With the trinomial lattice and the mean-tracking techniques, this paper presents the first bivariate lattice that guarantees valid probabilities. It also proves that any bivariate lattice for stock price and interest rate must grow superpolynomially if the interest rate model allows rates to grow superpolynomially.

Keywords: lattice, stochastic interest rate model, complexity

1 Introduction

The pricing of financial instruments with two or more state variables has been intensively studied. The added state variables to the underlying asset price can be stochastic volatility [8, 11] or stochastic interest rate [4, 14]. (The underlying asset will be assumed to be stock for convenience.) Unfortunately, many complex financial instruments with multiple state variables have no analytical formulas and thus must be priced by numerical methods like lattice. This paper presents a bivariate lattice method for models with stochastic stock prices and stochastic interest rates.

In the traditional approach to valuing derivatives, interest rates are often assumed to be constant. But the values of interest rate-sensitive securities such as callable bonds depend strongly on the interest rate, which does not stay constant in the real world. Hence it is critical for a general model to incorporate a stochastic interest rate component.

A stochastic interest rate model performs two tasks. First, it provides a stochastic process that defines future term structures. A term structure defines bond yields as a function of maturity. Second, the model should be consistent with the observed term structure. There are two approaches on modeling interest rates: the equilibrium model and the no-arbitrage model. Equilibrium models usually start with the assumption about economic variables and derive a process for the risk-free rate. Two of the difficulties facing equilibrium models are that (1) they usually require the estimation of the market price of risk and (2) they cannot fit the observed market term structure. Non-arbitrage models, in contrast, are designed to be consistent with the market term structure [10]. This paper focuses on non-arbitrage lognormal interest rate models, such as Black-Derman-Toy, Black-Karasinski, and Dothan models [2, 3, 7]. This class of interest rate models is assumed to follow the lognormal interest rate process. In this paper, we adopt Black-Derman-Toy (BDT) model, which is extensively used by practitioners, to explain the main idea of our bivariate lattice [1, 5, 13]. The popularity of lognormal models arises from the fact that negative interest rates are impossible.

Using the BDT model, Hung and Wang propose a bivariate binomial lattice for pricing convertible bonds [12]. A convertible bond entitles the holder to convert it into stocks. The lattice's size has a cubic growth rate. Chambers and Lu extend Hung and Wang's lattice by adding the correlation between stock and interest rate [6]. However, both approaches share the same problem of invalid transition probabilities. This is due to the fact that the BDT model allows the interest rate to grow superpolynomially, which makes the stock prices on the lattice with high interest rates unable to match the desired moments with valid probabilities.

This paper presents the first bivariate lattice for stock and interest rate that guarantees valid transition probabilities. Our bivariate lattice has two dimensions: the stock price dimension and the interest rate dimension. For the interest rate dimension, we construct a binomial interest rate lattice under the BDT model. We then develop a trinomial lattice for the stock price dimension with the help of mean-tracking techniques [16]. Furthermore, we prove that any bivariate lattice method (such as ours) for stock price and interest rate must grow superpolynomially if (1) the transition probabilities are guaranteed to be valid and (2) the interest rate model allows rates to grow superpolynomially such as the BDT model.

Our paper is organized as follows: The mathematical models are introduced in Section 2. We review how to construct a binomial lattice and a BDT interest rate lattice in Section 3. Section 3 also introduces the invalid transition probability problem. Section 4 describes the methodology for our proposed bivariate lattice. Section 5 proves the superpolynomial size of any bivariate lattice method for stock and interest rate if the interest rate model grows superpolynomially. Section 6 concludes the paper.

2 Modeling and Definitions

2.1 Continuous-Time Stock Price Dynamics

Define S_t as the stock price at time t. The risk-neutralized version of the stock price lognormal diffusion process is

$$\frac{dS}{S} = rdt + \sigma dz, \tag{1}$$

where r is the risk-free rate, σ is the volatility of the stock price process, and the random variable dz is a standard Brownian motion. Equation (1) has the solution,

$$S_t = S_0 e^{(r-0.5\sigma^2)t+\sigma z(t)}. \tag{2}$$

2.2 The Stochastic Interest Rate Model

In this paper, we use the Black-Derman-Toy (BDT) model for the interest rate dimension (but the general conclusion applies to all lognormal interest rate models). In the BDT model, the short rate r follows the stochastic process,

$$d \ln r = \theta(t)dt + \sigma_r(t)dz, \tag{3}$$

where $\theta(t)$ is a function of time calibrated to ensure that the model fits the market term structure, $\sigma_r(t)$ is a function of time and denotes the instantaneous standard deviation of the short rate, and dz is a standard Brownian motion.

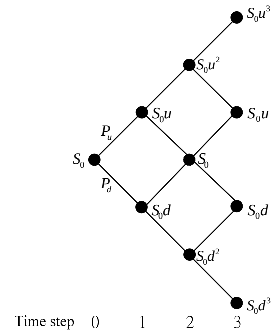

Figure 1: **The CRR Lattice.** The initial stock price is S_0. The upward and downward multiplicative factors for the stock price are u and d, respectively. The transition probabilities are P_u and $P_d = 1 - P_u$.

3 Preliminaries

3.1 The Binomial Lattice Model

A lattice partitions the time span from time 0 to time T into n equal time steps and specifies the stock price at each time step. The length of one time step Δt thus equals T/n. A 3-time-step Cox-Ross-Rubinstein (CRR) binomial lattice is illustrated in Fig. 1. At each time step, the stock price S can either make an up move to become Su with probability P_u or a down move to become Sd with probability $P_d \equiv 1 - P_u$. The relation

$$ud = 1 \tag{4}$$

is enforced by the CRR binomial lattice.

The mean (μ) and variance Var of $\ln(S_{t+\Delta t}/S_t)$ are derived from Eq. (2) thus:

$$\mu \equiv \left(r - 0.5\sigma^2\right) \Delta t, \tag{5}$$
$$\text{Var} \equiv \sigma^2 \Delta t. \tag{6}$$

By matching the mean and the variance of $\ln(S_{t+\Delta t}/S_t)$, the four parameters P_u, P_d, u, and d in the CRR lattice is

$$u = e^{\sigma\sqrt{\Delta t}}, \tag{7}$$

$$d = e^{-\sigma\sqrt{\Delta t}}, \tag{8}$$

$$P_u = \frac{e^{r\Delta t} - d}{u - d}, \tag{9}$$

$$P_d = \frac{e^{r\Delta t} - u}{d - u}. \tag{10}$$

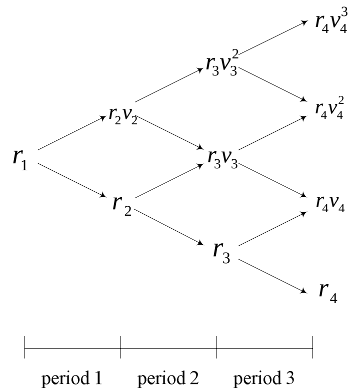

Figure 2: **The BDT Binomial Interest Rate Lattice.** The distribution converges to the lognormal distribution. The rates r_1, r_2, r_3, r_4 are called baseline rates.

The requirements $0 \leq P_u, P_d \leq 1$ can be met by suitably increasing n [15]. Moreover, the lattice converges to the continuous-time model as n increases. We remark that our results can generalize to any lattice with equal time step.

3.2 The BDT Binomial Interest Rate Lattice

An array of yields of zero-coupon bonds for numerous maturities and an array of short-rate volatilities for the same bonds are the input of the BDT model. From the term structure of yields, a procedure called calibration constructs a binomial lattice consistent with the term structure (see Fig. 2). In general, there are j possible rates in period j:

$$r_j, r_j v_j, r_j v_j^2, \cdots, r_j v_j^{j-1}, \tag{11}$$

where

$$v_j = e^{2\sigma_j \sqrt{\Delta t}} \tag{12}$$

is the multiplicative ratio for the rates in period j, r_j is called baseline rates, and σ_j is the annualized short-rate volatility in period j. The subscript j in σ_j is meant to emphasize that the short rate volatility may be time dependent. Each branch has a 50% chance of occurring in a risk-neutralized economy (i.e., the probability for each branch is 1/2).

3.3 The Invalid Transition Probability Problem

A few bivariate binomial lattices for pricing convertible bonds with the BDT interest rate model for the interest rate component have been proposed [6, 12]. If there is no correlation between the stock and interest rate, the transition probabilities for the stock price dimension are P_u and P_d given the prevailing interest rate r (recall Eqs. (9) and (10)). The no-arbitrage requirements $0 \leq P_u, P_d \leq 1$ is equivalent to

$$d < e^{r\Delta t} < u. \tag{13}$$

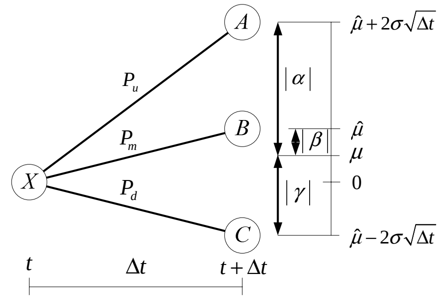

Figure 3: **The Trinomial Lattice.** The stock price can move from node X to node A with probability P_u, node B with probability P_m, and node C with probability P_d. Above, μ denotes the mean of the $s(X)$-log-price of $S_{t+\Delta t}$ (recall Eq. (5)). $\ln(s(X)/s(X)) = 0$, and $\ln(s(B)/s(X))$, $\ln(s(A)/s(X)$, and $\ln(s(C)/s(X))$ are $\hat{\mu}$, $\hat{\mu} + 2\sigma\sqrt{\Delta t}$, and $\hat{\mu} - 2\sigma\sqrt{\Delta t}$, respectively. The numbers β, α, and γ are defined in Eqs. (17)–(19). The length of the time step for the trinomial lattice is Δt.

In the above relations, u and d are independent of the prevailing interest rate r (recall Eqs. (7) and (8)). As the prevailing interest rate in the BDT model grows superpolynomially (which will be proved in Section 5), eventually inequalities (13) will break. When this happens, the probabilities generated by the CRR lattice will lie outside $[0, 1]$.

On the other hand, suppose the correlation (ρ) between the stock price and interest rate is not zero. The transition probabilities for the stock price dimension will become $P_u = p + \sqrt{p(1-p)}\rho$ and $P_d = 1 - P_u$, where $p = (e^{r\Delta t} - d)/(u - d)$. Obviously, the transition probabilities will be complex numbers if inequalities (13) do not hold.

4 Bivariate Lattice Construction

Our bivariate lattice has two dimensions: the stock price dimension and the interest rate dimension. For the stock price dimension, the foundation of our lattice is the CRR lattice mentioned in Section 3.1. To solve the invalid transition probability problem mentioned in Section 3.3, we use trinomial lattices to track the mean of the stock price.

Let the stock price of node Z be $s(Z)$ for convenience. Define the S-log-price of the stock price S' as $\ln(S'/S)$ and the log-distance between stock prices S and S' as $|\ln(S) - \ln(S')|$. Given a node X at time t and nodes at time $t + \Delta t$. (The log-distance between two adjacent nodes at time $t + \Delta t$ is $2\sigma\sqrt{\Delta t}$, which is the same as the CRR lattice). The mean (μ) and variance (Var) of the $s(X)$-log-price of $S_{t+\Delta t}$ can be obtained from Eqs. (5) and (6), respectively.

In Fig. 3, the node B whose $s(X)$-log-price ($\hat{\mu}$) is closest to μ among all the nodes at time $t + \Delta t$ will be the destination of the middle branch of the trinomial

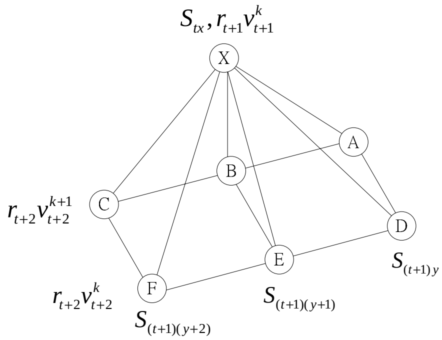

Figure 4: **The Concept of a Two-Dimensional Lattice.** The node X at time step t has six branches to nodes A, B, C, D, E, and F at time step $t+1$. Let S_{ij} denote the jth nodes at time step i. The stock price and the short rate at X are S_{tx} and $r_{t+1}v^k_{t+1}$, respectively. The stock price at nodes A and D is $S_{(t+1)y}$, that at nodes C and E is $S_{(t+1)(y+1)}$, and that at nodes D and F is $S_{(t+1)(y+2)}$. The interest rate at node A, B, C is $r_{t+2}v^{k+1}_{t+2}$ (upward), and that at nodes D, E, F is $r_{t+2}v^k_{t+2}$ (downward).

lattice (this is called meaning tracking). Note that the $s(X)$-log-price of the central node B will lie in the interval $(\,\mu - \sigma\sqrt{\Delta t}, \mu + \sigma\sqrt{\Delta t}\,]$ [16]. We then select two nodes A and C, which are adjacent to node B, and three transition probabilities P_u, P_m, and P_d for the node X can be obtained by matching the mean and variance of the $s(X)$-log-price of $S_{t+\Delta t}$.

The transition probabilities for the node X (i.e., P_u, P_m, P_d) can be derived by solving the following three equalities:

$$P_u\alpha + P_m\beta + P_d\gamma = 0, \tag{14}$$
$$P_u(\alpha)^2 + P_m(\beta)^2 + P_d(\gamma)^2 = \text{Var}, \tag{15}$$
$$P_u + P_m + P_d = 1, \tag{16}$$

where

$$\beta \equiv \hat{\mu} - \mu, \tag{17}$$
$$\alpha \equiv \hat{\mu} + 2\sigma\sqrt{\Delta t} - \mu = \beta + 2\sigma\sqrt{\Delta t}, \tag{18}$$
$$\gamma \equiv \hat{\mu} - 2\sigma\sqrt{\Delta t} - \mu = \beta - 2\sigma\sqrt{\Delta t}, \tag{19}$$
$$\hat{\mu} \equiv \ln(s(B)/s(X)). \tag{20}$$

(Note that $-\sigma\sqrt{\Delta t} < \beta \leq \sigma\sqrt{\Delta t}$.) The above procedure will yield valid probabilities [16].

For the interest rate dimension, we first construct a binomial lattice for the BDT model. The idea of our bivariate lattice is illustrated in Fig. 4. The node X at time t has 6 branches, to nodes A, B, C, D, E, and F at time $t + \Delta t$. Let S_{ij} denote the jth nodes at time step i. Assume that the stock price and the short rate at X are S_{tx} and $r_{t+1}v^k_{t+1}$, respectively. The stock price at nodes A and D is $S_{(t+1)y}$, that at nodes C and E is $S_{(t+1)(y+1)}$, and that at nodes D and F is $S_{(t+1)(y+2)}$. The interest rate at node A, B, C is $r_{t+2}v^{k+1}_{t+2}$ (upward), and that at nodes D, E, F is $r_{t+2}v^k_{t+2}$ (downward). In the above lattice construction, we simply assume that the interest rate process and the stock price process are independent, so the joint probabilities for the six branches can be obtained by multiplying the transition probabilities of the BDT binomial lattice (i.e., 1/2) and the trinomial lattice (i.e., P_u, P_m, and P_d). If the correlation between two processes are not zero, the joint probabilities can be obtain by using the orthogonalization technique [9].

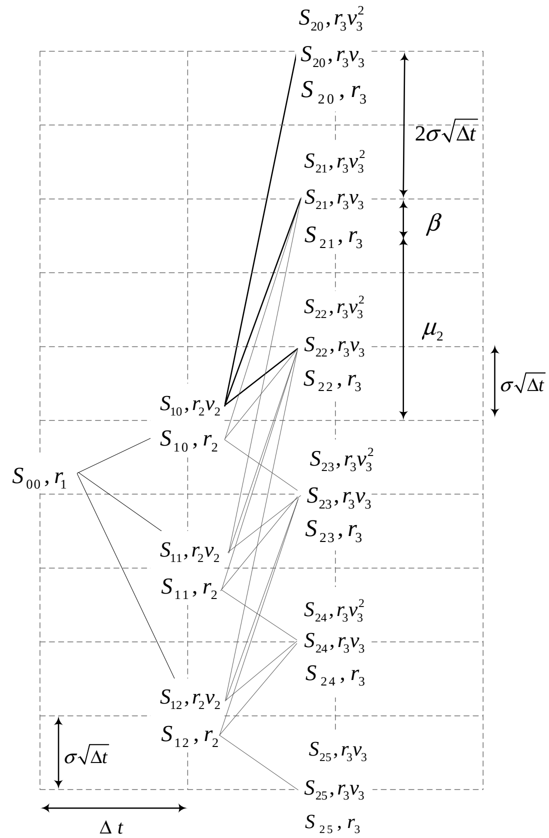

Figure 5: **The Size of the Bivariate Lattice.** The log-distance between two adjacent nodes is $2\sigma\sqrt{\Delta t}$. S_{ij} denotes the jth nodes at time i, and $\mu_2 = (r'_2v_2 - 0.5\sigma^2)\Delta t = r_2v_2 - 0.5\sigma^2\Delta t$ (recall Eq. (5)), where r'_j denotes the annualized baseline rate.

5 Complexity of the Bivariate Lattice

The size of the proposed bivariate lattice is proved in this section. Figure 5 gives a three-period bivariate lattice as an example and illustrates the concept of the bivariate lattice. In the figure, $\mu_2 = (r'_2v_2 - 0.5\sigma^2)\Delta t = r_2v_2 - 0.5\sigma^2\Delta t$ (see Eq. (5)) and

$$-\sigma\sqrt{\Delta t} \leq \beta < \sigma\sqrt{\Delta t}\,. \tag{21}$$

(Note that r'_j denotes the annualized baseline rates.) To investigate the complexity of the size of the bivariate lattice, we prove that the node count at each time grows superpolynomially as n goes to infinity.

We fix the maturity $T = 1$ for simplicity. Let $d(\ell)$ denote the number of nodes in the stock price dimension at time step ℓ and μ_j denote the log-mean of the stock price at time j with the biggest interest rate at time $j-1$ (i.e., $\mu_j = r_j v_j^{j-1} - 0.5\sigma^2/n = r_j e^{2(j-1)\sigma_j/\sqrt{n}} - 0.5\sigma^2/n$) for $j = 1, 2, \cdots, n+1$. Figure 5 shows that $d(0) = 1$, $d(1) = 3$, and

$$d(2) = d(1) + 1 + \frac{\mu_2 + \beta + 2\sigma\frac{1}{\sqrt{n}} - \sigma\frac{1}{\sqrt{n}}}{2\sigma\frac{1}{\sqrt{n}}}.$$

By the fact $-\sigma\frac{1}{\sqrt{n}} < \beta \leq \sigma\frac{1}{\sqrt{n}}$ by inequalities (21), we know that

$$d(1) + 1 + \frac{\mu_2}{2\sigma\frac{1}{\sqrt{n}}} \leq d(2) \leq d(1) + 1 + \frac{\mu_2}{2\sigma\frac{1}{\sqrt{n}}} + 1.$$

By induction, $d(l+1)$ satisfies the following recurrence relation:

$$d(1)+\ell+\sum_{j=2}^{\ell+1}\frac{\mu_j}{2\sigma\frac{1}{\sqrt{n}}} \leq d(\ell+1) \leq d(1)+2\ell+\sum_{j=2}^{\ell+1}\frac{\mu_j}{2\sigma\frac{1}{\sqrt{n}}}. \quad (22)$$

Consequently, the total node count in the stock price dimension is

$$\sum_{j=0}^{n+1} d(j). \quad (23)$$

From Eqs. (22) and (23), we know that any bivariate lattice method for stock and interest rate must grow superpolynomially if the interest rate (or μ_j) grows superpolynomially (, to which we turn below).

Next, we prove that the growth rate of μ_j is superpolynomial in the BDT model. We first prove the superpolynomial growth rate of the biggest interest rate at time $j-1$, $r_j v_j^{j-1} = r_j e^{2(j-1)\sigma_j\frac{1}{\sqrt{n}}}$. Let f_j denote the forward rate in period j. Build a binomial interest rate lattice with the baseline rates r_j^* set by

$$r_j^* = \left(\frac{2}{1+v_j}\right)^{j-1} f_j. \quad (24)$$

Theorem 5.1 *The binomial interest rate lattice constructed with Eq. (24) overestimates the prices of the benchmark securities in the presence of volatilities. The conclusion is independent of whether the volatility structure is matched.*

Proof: The details of the proof is presented in [15] . ■

Theorem 5.2 *The binomial interest rate lattice constructed with Eq. (24) by using the forward rate $v_j f_j$ instead of f_j underestimates the prices of the benchmark securities in the presence of volatilities. The conclusion is independent of whether the volatility structure is matched.*

Proof: The details of the proof are available from the authors upon request. ■

According to Theorems 5.1 and 5.2, we have

$$r_j^* < r_j < r_j^* v_j. \quad (25)$$

Then

$$\begin{aligned}
r_j v_j^{j-1} &< r_j^* v_j^j && (26)\\
&= \left(\frac{2}{1+v_j}\right)^{j-1} f_j v_j^j && (27)\\
&= \left(\frac{2v_j}{1+v_j}\right)^{j-1} f_j v_j, && (28)\\
r_j v_j^{j-1} &> r_j^* v_j^{j-1} && (29)\\
&= \left(\frac{2}{1+v_j}\right)^{j-1} f_j v_j^{j-1} && (30)\\
&= \left(\frac{2v_j}{1+v_j}\right)^{j-1} f_j. && (31)
\end{aligned}$$

Let

$$A(n,j) = \left(\frac{2v_{j+1}}{1+v_{j+1}}\right)^j = \left(\frac{2e^{2\sigma_{j+1}\frac{1}{\sqrt{n}}}}{1+e^{2\sigma_{j+1}\frac{1}{\sqrt{n}}}}\right)^j. \quad (32)$$

Because $1 + e^{2\sigma_{j+1}/\sqrt{n}} > 2$,

$$A(n,j) < \left(\frac{2e^{2\sigma_{j+1}\frac{1}{\sqrt{n}}}}{2}\right)^j = e^{2\sigma_{j+1}\frac{j}{\sqrt{n}}}. \quad (33)$$

Therefore, $A(n,j) = e^{O(j/\sqrt{n})}$. On the other hand, for n sufficiently large, $2 \times 2^{2\sigma_{j+1}\sqrt{\frac{1}{n}}} \geq 1 + e^{2\sigma_{j+1}\sqrt{\frac{1}{n}}}$. So

$$A(n,j) \geq \left(\frac{2e^{2\sigma_{j+1}\frac{1}{\sqrt{n}}}}{2 \times 2^{2\sigma_{j+1}\sqrt{\frac{1}{n}}}}\right)^j = \left(\frac{e}{2}\right)^{2\sigma_{j+1}\frac{j}{\sqrt{n}}}. \quad (34)$$

As a result, $A(n,j) = (e/2)^{\Omega(j/\sqrt{n})}$ for n sufficiently large.

According to Eqs. (26), (29), (33), and (34), as n goes to infinity, the upper and lower bounds of $r_j v_j^{j-1}$ are

$$A(n, j-1)v_j < r_j v_j^{j-1} < A(n, j-1) f_j v_j, \quad (35)$$

From inequalities (35), we know that $r_j v_j^{j-1}$ grows superpolynomially, and equivalently, μ_j grows superpolynomially (recall $\mu_j = r_j v_j^{j-1} - 0.5\sigma^2/n$). Therefore, the size of the bivariate lattice must grow superpolynomially in the BDT model.

6 Conclusions

This paper presents the first bivariate lattice method to solve the invalid transition probability problem by using the trinomial lattice and mean-tracking techniques. Furthermore, we prove that any bivariate lattice method for stock price and interest rate must grow superpolynomially if (1) the transition probabilities are guaranteed to be valid and (2) the interest rate model allows rates to grow superpolynomially such as the BDT model.

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