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Self-reference and recursive forms

Louis H. Kauffman

Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, Chicago, IL 60680, USA

Introduction

The purpose of this essay is to sketch a picture of the connections between the concept self-reference and important aspects of mathematical and cybernetic thinking.

In order to accomplish this task, we begin with a very simple discussion of the meaning of self-reference and then let this unfold into many ideas. Not surprisingly, we encounter wholes and parts, distinctions, pointers and indications, local–global, circulation, feedback, recursion, invariance, self-similarity, re-entry of forms, paradox, and strange loops. But we also find topology, knots and weaves, fractal and recursive forms, infinity, curvature and imaginary numbers! A panoply of fundamental mathematical and physical ideas relating directly to the central turn of self-reference.

What is self-reference?

At least one distinction is involved in the presence of self-reference.
The self appears, and an indication of that self that can be seen as separate from the self.
Any distinction involves the self-reference of 'the one who distinguishes'.
Therefore, self-reference and the idea of distinction are inseparable (hence conceptually identical).

We explore self-reference by examining what appear to us as distinctions. Through experiencing self-reference, we come to understand the possibility of distinguishing.

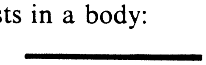
Mark

A mark or sign intended as an indicator is self-referential. The self is the whole space including the mark and the observer. But the mark points, in the first place, to its own location, and in this process becomes a locus of reference. The mark refers to itself. The whole refers to itself through the mark.

Pointing

Pointing is represented by an arrow.

The anatomy of the arrow consists in a body:



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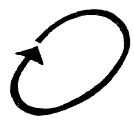
and a barb (or mark, or pointer):

▶

Together they accomplish directionality and indicate the possibility of movement from locale to locale:

When we turn the arrow on itself we achieve self-reference with the whole as the arrow itself, and the barb and tail as 'parts':

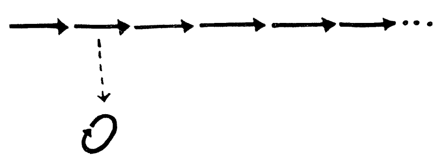


The self-pointing arrow becomes self-referential only through the agreement of an observer. Thus, it achieves self-reference through primary (primordial) self-reference. At the same time, the self-pointing arrow is a symbol for the condition of observation in which the self appears to divide itself into that which sees and that which is seen. Thus do barb and tail appear separate although they are joined.

Circulation

The self-pointing arrow forms a directed circle. This invites travel around the cycle. From observer to observed to observer to observed to . . .

This circulation binds parts back into the whole, and it has the geometry of what we might call the *unidirectional circular unfolding*:



In this unfolding, the individual arrow bent around to form a circle has become an ordered infinity of arrows, each representing a particular trip around the cycle. In total, these arrows, set tip to tail, create a directed line pointing off to infinity.

Self-reference is the infinite in finite guise!

The unfolding itself holds the simple secrets of any infinite form. These are *invariance* and *self-similarity*. Invariance manifests itself through the circumstance that after many turns of the wheel it is possible to lose count—to lose the sense of difference between corresponding places on different cycles. Imagine winding down an infinite stairwell whose features repeat and repeat. How long has it been? Will this process end?

In the simple example of an infinite string of arrows, the invariance can be indicated symbolically as below:

$$a = >>>>>>>>>>>>>>>>>>> \ldots$$

Hence,

$$a = >a \quad \text{when } a = >>>>>>>>>> \ldots$$

The three dots indicate the infinity of arrows to the right. The 'infinite arrow' a is *form invariant* under adding any single arrow on the left.

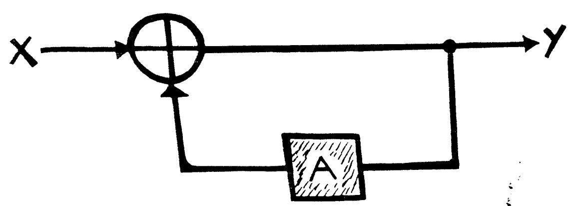
Note that the equation $a = \; > a$ is also an expression of self-reference in that it describes a in terms of itself. And within the present context, this is sufficient for the reproduction of a as an unending process.

Self-similarity is embodied in the expressed fact that a has a copy of itself within itself. This is another reading of the equation $a = \; > a$. How is this formal self-similarity related to our intuition of self-within-self through introspection? I suggest that in form these circumstances are identical. It is in moving through the cycle *and* seeing the invariance that we come to a reflection of the self. But note that this personal process involves the non-mechanical aspect of integration of the parts into a whole. I say non-mechanical because there is no way to formalize the entire circumstance of human self-reference in a system of symbols devoid of an observer. But who or what is the observer?

Feedback

The equation $a = \; > a$ is also an abstract instance of feedback in that it shows how the form is returned into itself. This is not yet feedback with a positive or negative sense attached, but only the abstract pattern of the process.

To see the analogy more clearly, consider the classical situation of a transfer function:



$$Y = AY + X$$
$$Y - AY = X$$
$$Y(1 - A) = X$$
$$Y = [1/(1 - A)]X$$
$$Y = TX, \quad T = [1/(1 - A)]$$
$$[1/(1 - A)] = 1 + A + AA + AAA + AAAA + \ldots$$

Here the self-referential equation $Y = AY + X$ describes the balance at stability of the feedback system with return transfer function A. This equation is regarded as algebraic as well as descriptive, and hence can be formally solved for Y. This solution gives the transfer function $T = [1/(1 - A)]$ for the whole system. Upon expanding T as a power series

$$T = 1 + A + AA + AAA + AAAA + \ldots,$$

we come again to the unidirectional circular unfolding. Here, the terms 1, A, AA, ... each represent the contribution to Y by feedback from 1, 2, 3, ... circulations of the loop. Thus is the abstract formalism of self-reference embodied in the everyday calculations of engineering practice.

But it is worthwhile to stay with this example longer and to examine its structure further. For there is more than one level of language operating in the transfer function calculation. This is embodied in the meaning of the power series

$$T = 1 + A + AA + AAA + AAAA + \ldots$$

On the one hand, the identity $T = 1/(1 - A)$ is *formally* correct in that

$$T = (1 + A + AA + AAA + \ldots) = 1 + A(1 + A + AA + \ldots)$$

$$T = 1 + AT$$

as (infinite) strings under the conventions of algebraic manipulation ($A1 = A$, use of parentheses, . . .).

On the formal level, the identities $T = 1 + AT$ and $a = \, >a$ are structurally identical. But the feedback equation $T = 1 + AT$ is also interpreted numerically and in terms of the corresponding numerical process. Here, another language intervenes, and it is the beauty of this mathematics that it works at both levels. This aspect of harmony among different levels of discourse is present from the beginning in our discussion of self-reference, but the initial harmony of a model with the actual experience of self and reference is very open. It is open both to personal taste and to the possibility of new insight. Thus, the possibility of harmony in mathematics between the formal and the analytical is also an open arena for creativity.

These last remarks may seem to have strayed away from the theme of feedback that opened this section. However, I urge the reader to look into the technical and complex examples of feedback from his/her experience for the wisdom that they potentially contain about reference and the self. It is in this richness of many languages that the reality of the process lives. And yet self-reference is no more complicated than a self that can refer!

Re-entry

Another way to view self-similarity is through the concept of *re-entry*. This view was first made explicit by Spencer-Brown (1972; 1979, chapters 11 and 12). Here, one encounters the idea of a form that re-enters its own indicational space:

> We take as given the idea of distinction and the idea of indication, and that one cannot make an indication without drawing a distinction. We take therefore the form of distinction for the form (Spencer-Brown 1972; 1979, chapter 1).

A space is severed or taken apart. Form appears in the process, and the form appears to enter into or re-enter the very space that generated it. At first it may seem paradoxical that the original form, the form of the severance, partaking of the whole space and its cleft could somehow re-enter this space to be seen as part of it. This is the movement of language from partaking to part. And in that first movement, the part is not yet separate from the whole. It partakes of it and it is it. Yet inevitably, there comes through the possibility of seeing anything at all the possibility of seeing a separate part. And so does the part become divided from the whole while still enfolded within it.

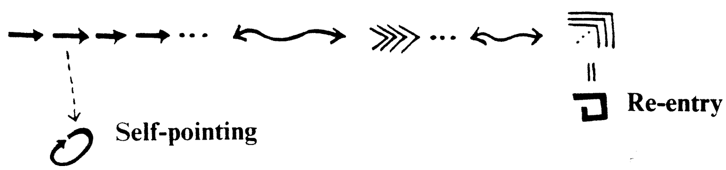
This description of the dance of part and whole *is* the form of distinction, and the description is itself a part and partner in the re-entry. So it is with the dialog of whole and part in our description of self-reference.

In the unidirectional unfolding of the self-reference, we found a self-similar infinite arrow $>>>>>>>>>> \ldots$ and its descriptive form $a = \, >a$. Here, we are literally describing a form that sits within itself. The movement to form within is accomplished with the geometry of a shift (of one unit) to the right. This shift corresponds to one turn about the underlying cycle, and hence in our interpretation, to the movement from part to whole to part again. It is fascinating to see how the general philosophical ideas about re-entry and self-reference fit these simple mathematical models.

The re-entry shown in the equation $a = \, >a$ has been illustrated in the language of Spencer-Brown by Varela (1975) through the symbolism of the *re-entering mark* (see below).

Here, the infinite arrow has become an infinite descending chain of Spencer-Brown marks, and the re-entry is indicated by an under-turning hook in the re-entering form. This version of re-entry, like the self-pointing arrow, has the advantage of a rich field of associations. It reminds us of the symbolism of the world-snake eating its own tail, and of an abstraction of the idea of feedback to a notion of process moving through and being form. A non-numerical feedback of meaning and pattern and symbol and sign, which has slipped beyond linear reason into a fugue of thought that boils up out of the core of nothingness.

Here, we articulate a distinction between self-pointing form and re-entering form. In the mathematics, the re-entering form is seen as the unidirectional unfolding of the self-pointing form. Nevertheless, both self-pointing and re-entering are encapsulated in the concept of self-reference!



Speaking of itself

Self-reference is often associated with language that is self-referring, and through this with paradoxes, puzzles and strange turns of speech. Some examples:

This statement is false.
This statement is true.
One of the sentences above is false.
You are reading this sentence.
I only exist when you read me.
This sentence contains five words.

One can go on and on with this sort of thing. The point I wish to discuss in this section can be indicated by the question: *is self-reference in language a form of re-entry*?

In every case, there is a pointing of the sentence to itself (or to a part thereof). Usually, this takes the form of a part of the sentence acting as pointer toward the whole. In grammatical form, there is little difference between this and an external reference. Compare the sentences below:

This sentence speaks the truth.
This woman speaks the truth.

Sometimes the strangeness results from a reference usually taken as being external which is suddenly seen as turning back on the sentence itself:

Do you read me out loud?

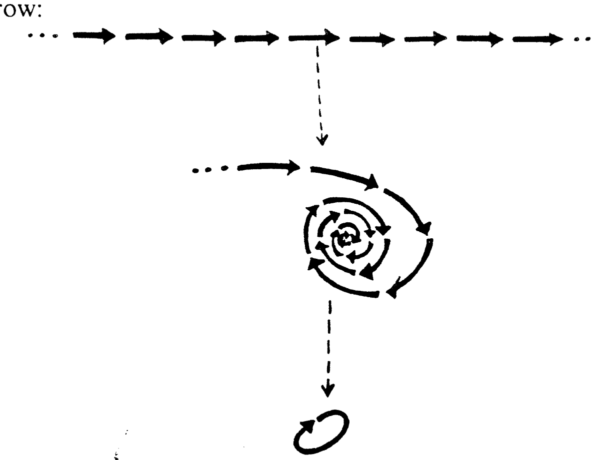
In no case is there a re-entry in the strict sense of a form inside a form. This requires infinite regress, as in the arrow form a = > > > > > > > . . . Nevertheless, such regress can be indicated by the self-referential sentence and can lead the speaker/reader/performer into the process. (The liar paradox can create an oscillation between true and false).

True re-entry can never occur on the page of symbols. It requires presence, the presence of text and context, of reader and what can be read. In this sense, re-entry in language and the referring of the self are one.

Feedback and symmetry

The concepts of re-entry and symmetry are extraordinarily close. It may seem surprising at first to think that the beautiful symmetry of a pattern of Alhambra knotwork (see Fig. 1) can have anything to do with feedback and cybernetics, yet this is indeed the case!

To begin to see this relationship, let us consider the *bi-directional unfolding* of the self-pointing arrow:



Here, we have allowed the arrows to go end-to-end in two directions. In condensed form we have

$$b \;=\; \ldots >>>>>>>>>>>>>>>>> \ldots$$

The infinite form b has a left–right translational symmetry. If T is the operation of shifting one (arrow) unit to the right, then we can say

$$T(b) \;=\; b.$$

This is invariance under a symmetry operation, the same sort of invariance seen every day in looking at frieze patterns and decorated borders.

Recall that the mathematical concept of symmetry involves the presence of a geometrical operation T such as a rotation, reflection or translation, such that the form F is invariant under the operation of the transformation. Thus, the human figure is approximately symmetrical via mirror reflection in a mirror that bisects the body from side to side.

The abstract idea of invariance under transformation is shared in the formalism of both symmetry and self-similarity. Thus, we have the comparison of the equations

$$a \;=\; >a$$

$$b \;=\; T(b)$$

The forms a and b are each invariant under a certain transformation, but we regard T as a symmetry since it does not change the shape or size of b, but in moving it, it matches b to itself.

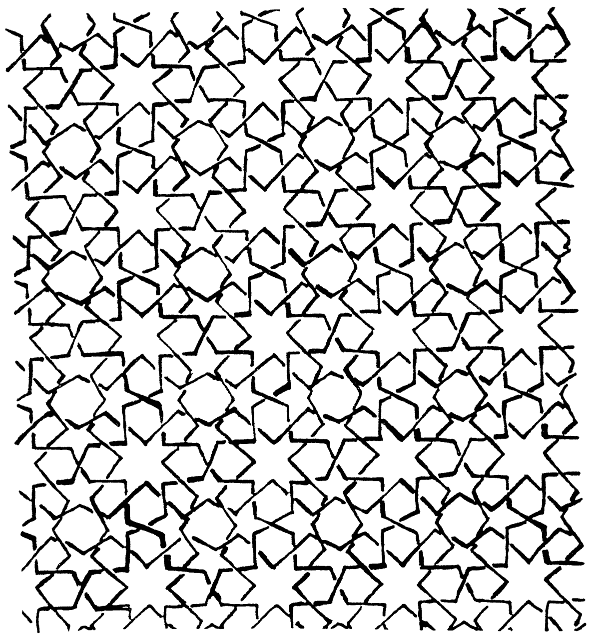


Fig. 1. From the Alhambra. (Also on an inner door in gold trim in the church La Seo de Zaragoza, Zaragoza, Spain)

I believe that is appropriate to consider the self-similarity $a = >a$ as a kind of generalized symmetry. This can be formalized, but we shall not do that here. It is also possible to continue the analysis of self-referential forms in terms of symmetries. The appropriate mathematical concept for this is the *covering space*, a notion familiar to topologists (see Munkres, 1975). What we have called the bi-directional unfolding is the well-known way in which the real line covers the circle via the complex exponential mapping. A fancier covering space is shown in Figure 2. This is the universal covering space of the figure eight. It has vertical and horizontal (topological) symmetry directions.

It is important to realize that the concept of feedback is actually part of a powerful generality that includes symmetry and invariance under groups of symmetries. In this framework, for example, recent work on fourier analysis in geometrically periodic neural nets may be seen as a fugue of self-reference in two forms—feedback and symmetry (Varela & Jorge, in preparation).

The concept of feedback is normally associated with a system that can be decomposed into inputs and outputs. The feedback is a 'feeding back' of (a transform of) the output into the input. This results in circulatory and recursive behavior patterns in the system. Under appropriate conditions, the recursion settles down to a steady form of interaction.

Thus, the driver of a car is part of a feedback loop involving the car, the environment and his/her steering. This, however, is a closed loop, and one might see that we have made an arbitrary, but culturally understandable, choice in splitting the loop as inputs to and outputs from the human driver. In fact, the situation of human beings is such that there is no such splitting in the act of driving. The driver knows that his being in the car is a simultaneity with the operation of the vehicle. It is only in the circumstance of a breakdown, such as on particularly icy roads, or an oncoming speeding car in the same lane, that there occurs an apparent separation of driver and automobile. Under normal

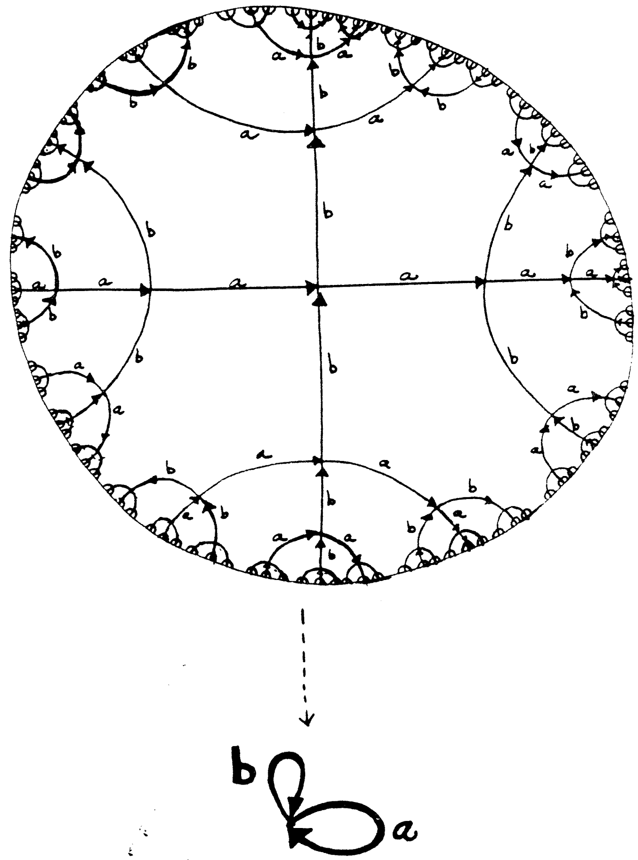


Fig. 2. Unfolding the figure eight. (Horizontal arrows lift a. *Vertical arrows lift* b*)*

circumstances, the driver and the vehicle are apparent parts of a whole system that has properties going beyond either one of them.

Thus, feedback is a particular way of viewing a system that has behavior, by looking at it in terms of an input–output model and/or in terms of a model involving circular paths of energy and information that *can* be cut to create apparent inputs and outputs.

That there is a possibility for modelling in terms of input–output function does not mean that this view is fundamental. But it is very powerful. And its effectivity can be enhanced through the understanding that there may be more than one way to cut the loops!

In mathematical problems of a topological type, this point is particularly apparent, as an example from knot theory given later in this essay will show. In any practical situation, we simply make do with those entry points into the system that are available to us.

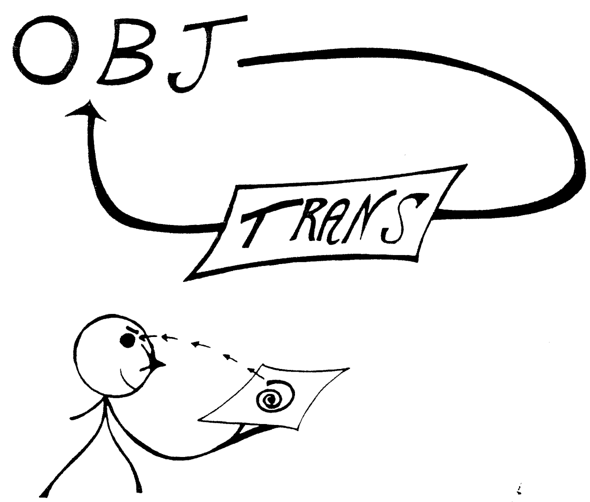
Nevertheless, it is important to be quite aware of our assumptions. Thus, even the use of the words 'entry point' presupposes a model of transmitting 'something' into 'something else'. What is transmitted in the awareness of the self?

The reason I bring up these epistemological considerations in regard to feedback lies in the analogy between feedback and symmetry. In symmetry, we have an 'object' (possibly mathematical) OBJ and a transformation TRANS (perhaps a rotation of the space in which the object is embedded), and we say that TRANS is a *symmetry* of OBJ if the equation

$$\text{TRANS(OBJ)} = \text{OBJ}$$

holds. One then goes on to classify all the transformations leaving the object invariant in this manner, thereby arriving at symmetry groups and a rich structure. In quantum physics, the concept of the particle as the object has been all but superseded by the concept symmetry group.

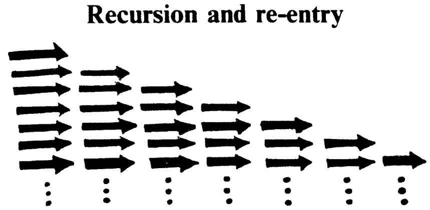
The desire to call symmetry a something like feedback arises when making a diagram like the following one.



Here, the OBJ seems to have been fed back through TRANS to be a stable condition of the feedback! Thus, we arrive at the notion of a form being fed back through and invariant under an abstract, perceptual, or mathematical transformation. If we allow that which is fed through the feedback loop to be a pattern or even a concept then we have arrived at a realm where feedback and symmetry exist hand-in-hand. Such an arena does indeed occur in cognition. And it is in this place that both neurophysiological research (Varela & Jorge, in preparation) and psychological work (Lefebvre, 1972, 1982) share common cybernetic patterns.

Thus, without complete formalization, we have pointed to a domain where re-entry, self-similarity, and feedback are all different expressions or exfoliations of the structure of awareness. Primary awareness opens the possibility of the self.

Recursion and re-entry

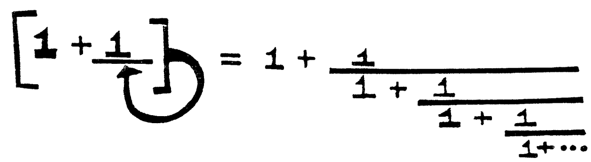


Recursion has been present from the very beginning of the discussion. For a specific image, consider the following sequence:

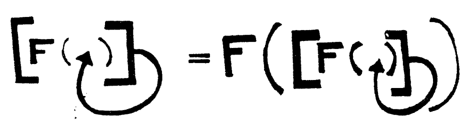
$$
\begin{array}{l}
> \\
> > \\
> > > \\
> > > > \\
> > > > > \\
\ldots
\end{array}
$$

Here, we grow approximations to the form $> > > > > > > > > > > > > > \ldots$ by the recursion that adds a new arrow at each step. In this sense, the equation $a = > a$ can be regarded as a shorthand form of this process. Any description via re-entry becomes a program for the generation of a self-similar form. (See Kauffman & Varela, 1980 for more details and further references.)

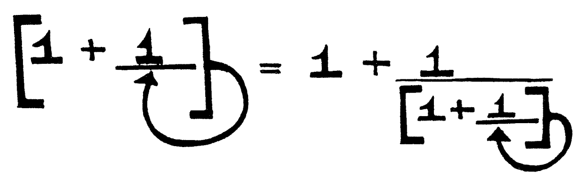
For example, consider the re-entry form:



Here, we use the arrow to indicate the point of re-entry, a possible nuance of meaning for the original self-pointing arrow. The notation with brackets and arrow line pointing to specific locations of re-entry is used recursively with the *first convention: the sign of enclosure is erased.*



Thus, in the case of the continued fraction we have the recursion



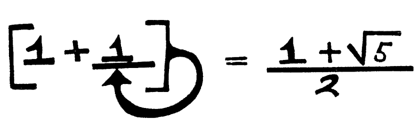
and corresponding sequence of numerical approximations

$$
1, \quad 1 + \tfrac{1}{1}, \quad 1 + \frac{1}{1 + \frac{1}{1}}, \quad 1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{1}}}
$$

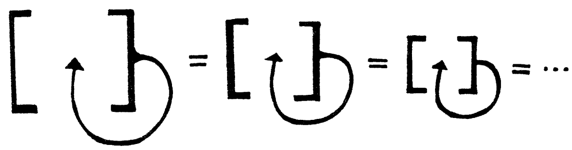
$$
1 \qquad \tfrac{3}{2} \qquad \tfrac{5}{3} \qquad \tfrac{8}{5}
$$

(1, 1, 2, 3, 5, 8, 13, . . . Fibonacci series)

and in the limit

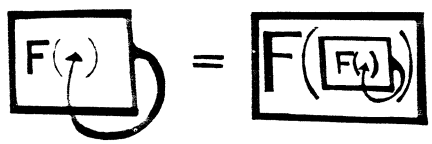


It is amusing to note that in the case of the first convention *when there is no internal structure to the re-entry, the recursion step leaves the form invariant*:

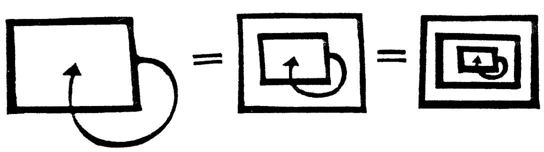


This is a mathematical image of the condition of pure self-reference (see Merrell-Wolff, 1976).

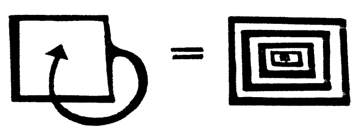
In the *second convention, the sign of enclosure is not erased*. Hence,



Thus,



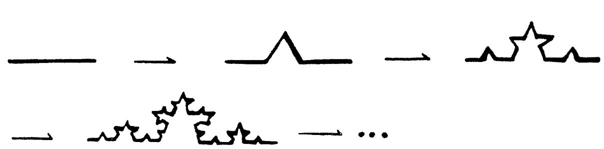
and in the limit

 (ad infinitum).

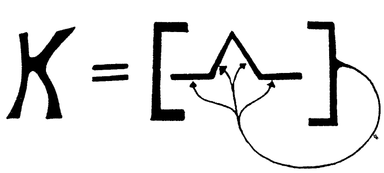
This form of re-entry is exactly the form of the re-entering mark discussed earlier in the essay. Once again, we see a splitting related to how one views self-reference in a recursive framework.

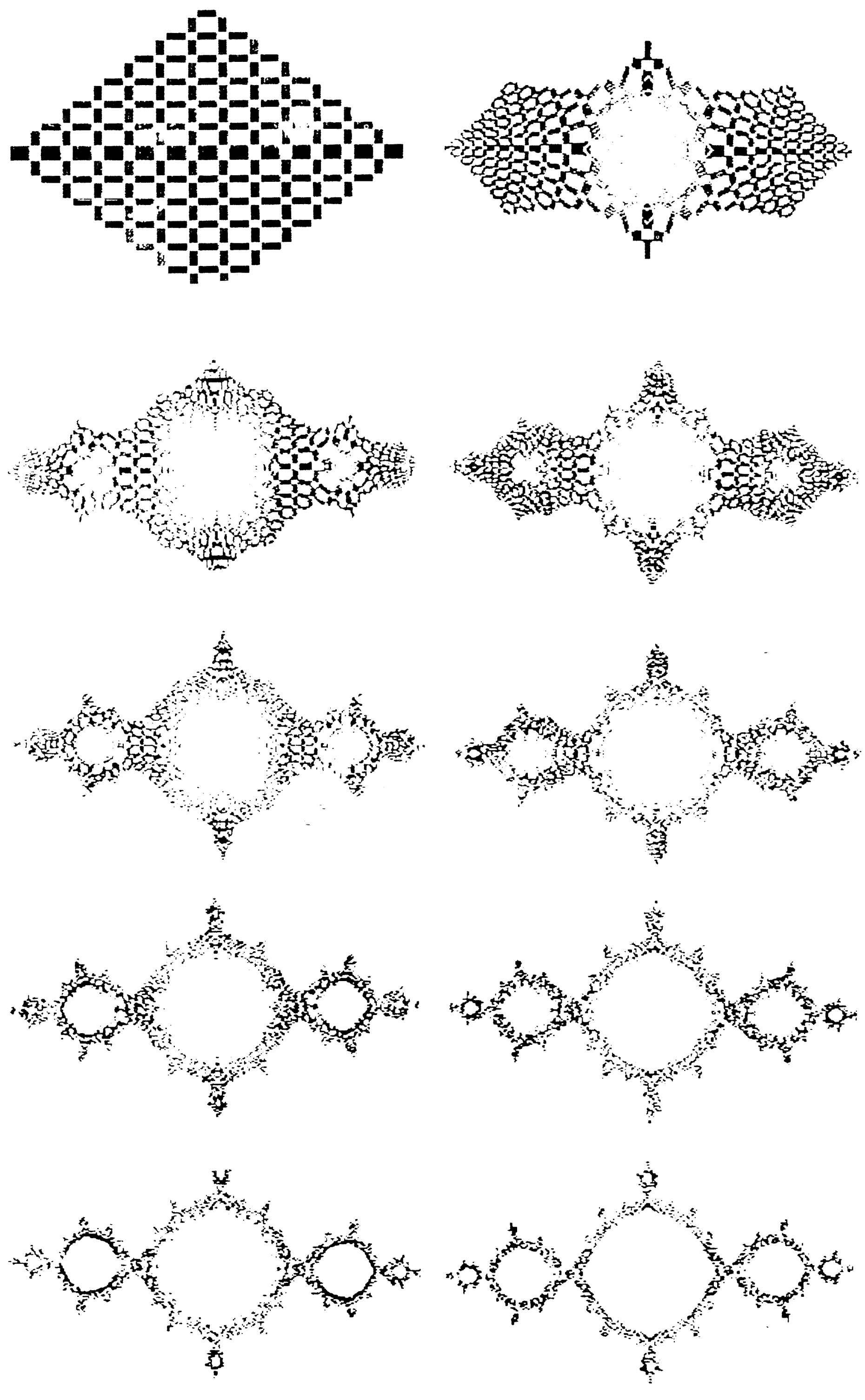
Recursive forms and fractals

By allowing more geometry into the abstract realm of the re-entry forms, we find ourselves in a rich spatial arena. As a first example, consider the well-known Koch curve (Mandelbrot, 1982). This is a self-similar curve obtained through the geometric recursion:



Consequently, the Koch curve may be described as a geometrical re-entry form





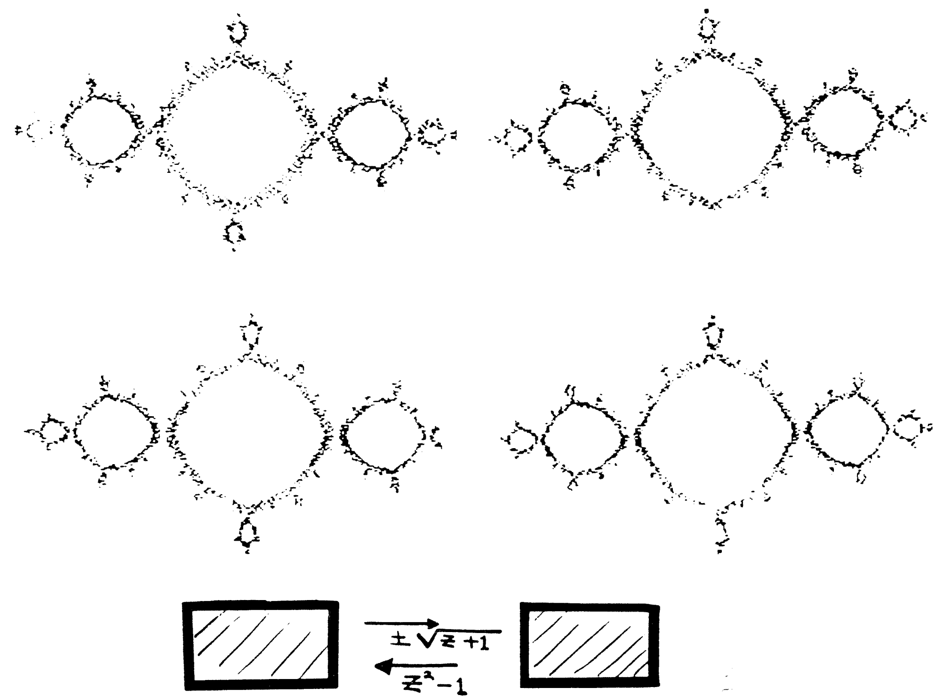
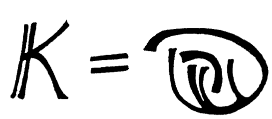


Fig. 3. An example of the Julia set where K takes the value 1

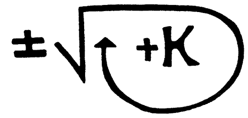
and hence is related to the abstract re-entry form:



Here, we have indicated the change of level of the inner part of the Koch curve by using extra enclosures in the abstract form. It is possible to carry this analysis further and investigate the boundary between the abstract forms and their geometric realizations.

As a second example, consider the function $F(z) = \pm SQRT(z + K)$, where K is a complex number, z is a complex variable, and $\pm$ denotes both of the square roots of the associated complex number $z + K$. By regarding this function as a mapping of the complex plane to itself and iterating it on a test pattern P, we form the sequence of images P, $F(P)$, $F[F(P)]$, . . . The images converge (set-wise) to a specific image I such that $F(I) = I$. This image is an example of a so-called Julia set (Mandelbrot, 1982), and it embodies an intricate structure of self-similarity that depends upon the choice of constant K. (See Fig. 3 for an illustration where K takes the value 1.)

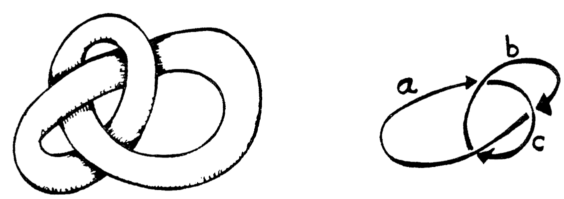
Here, the re-entry form related to



is both extraordinary and beautiful, and it leads to deep mathematics.

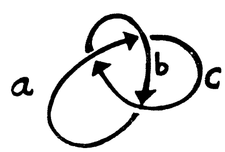
Knots and topology

Another movement into geometry occurs through the formal theory of knots. A knot is a mathematical model of knotted rope in space. For technical reasons, it is convenient to assume that the knotting occurs on a closed loop, as in the diagram for the trefoil knot shown below:

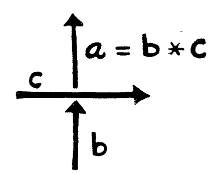


In this diagram, we have shown a semi-realistic drawing juxtaposed with the schematic drawing that is used as notation in formal knot theory. The schematic diagram resolves into a collection of separate but related segments, here labelled a, b, and c. These segments result from the particular projection of the knot into the plane that is indicated by this picture.

A knot is a whole form in three-dimensional space. It becomes cut up into parts through the process of projection. In order to language the knot, to describe it, one must cut it up into circularly interrelated parts. These parts then become referenced in circular patterns of definition:



$$a = b * c$$
$$b = c * a$$
$$c = a * b$$

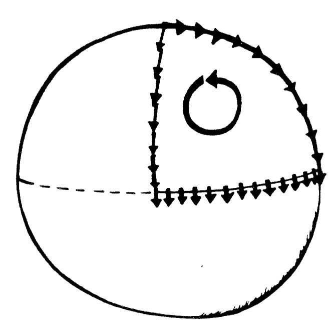


Deformation of the knot leads to a new projection and to a different self-referential description. Nevertheless, all these viewpoints derive from a single topological form, whose decomposition is an artifact of the process of projection.

In topology, we have a paradigm case for the movement from a whole to part(s). The ability to purvey the whole from a geometric point of view is unique to this mathematics and may provide intuition for proper treatment of self-reference in language and cybernetics where global structure is not so apparent.

There are other mathematical themes of relevance to the articulation of self-reference. Closest to re-entry is the concept of curvature seen as a measure of difference generated by travel around a closed loop (Einstein, 1956). The difference may be a deviation from parallelism or the change in phase of a gauge field, but the general theme of measurement of the state of re-entry pervades both mathematics and physics.

For a concrete illustration of the effects of curvature, consider the trip illustrated on the globe below:



Here, we move downward from the north pole to the equator, then along the equator for one-quarter turn, then back up a line of longitude to the north pole. If we always translate an arrow parallel to itself throughout the journey, we expect to arrive back at the pole with the arrow oriented just as it was when we left. But it has been turned by ninety degrees! This is the measure of curvature through difference seen at re-entry.

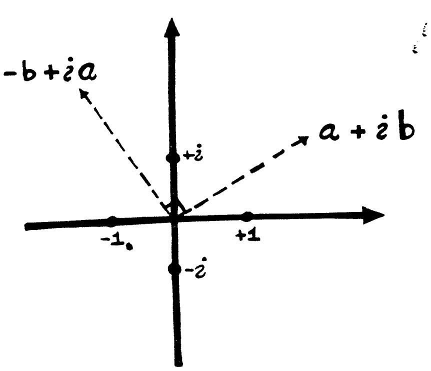
As a parting example, consider the Mobius band. The global twist makes the first return after one circulation of the band reverse orientation.

These are all phenomena in a field that could be called the *topology of self-reference.*

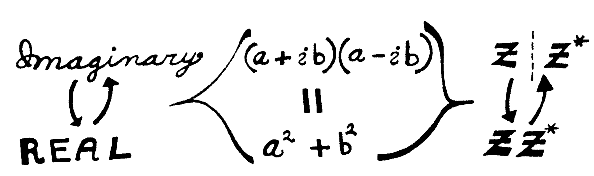
The imaginary

Recall that in mathematics, one uses *complex numbers* of the form $a + bi$, where i denotes a 'square root of minus one'. The number i extends the concept of real numbers, allowing many useful applications and relationships.

At the outset, complex numbers were a bit suspect, since i seemed paradoxical. Later, it was understood that the complex numbers could be interpreted as points in the co-ordinate plane, so that $a + bi$ meant the point obtained by going east or west of the origin by a units (direction depending upon the sign of a) and then north or south by b units. Thus, i itself resides in this (complex) plane at a point one unit distance perpendicular to the horizontal axis. Furthermore, it was understood that the operation of multiplication by i resulted in a ninety-degree turn in this plane. Thus $ii = -1$ can be read to mean: *two ninety-degree, counterclockwise rotations result in a rotation of one hundred and eighty degrees.* On the real line, a number and its negative lie at one hundred and eighty degrees away from one another. This geometric interpretation of the imaginary number i explains the apparent mystery of i in fruitful geometric terms.



Imaginary numbers are useful because they form a domain that can be entered from reality (real numbers) and within which insights and structural moves of a new order are available. And one may re-enter the real domain from the imaginary domain. Each supports the other.



In this way, it is possible to solve equations (historically the real roots of a cubic equation) that could not be solved directly. The method continues to be of use in solving differential equations. For example, the imaginary constant i is inseparable from the framework of quantum physics, where re-entry from the complex domain is accomplished for the wave function ψ by multiplying it by its complex conjugate. This forms the real probability density $\psi * \psi$. This real function describes numerically the possibility of observation of a field (particle/wave), while the complex form provides the formalism for modelling interference and evolution.

There is a close connection between our formalisms for self-reference and the square root of minus one. I will only sketch this here. For further information, see Kauffman (1978, 1985, 1986), Kauffman & Varela (1980), Spencer-Brown (1972, 1979), and Varela (1975).

First, there is the commonality of apparent paradox. The imaginary number i is supposed to satisfy $T(i) = i$, where $T(x) = -1/x$. The re-entering mark is asked to satisfy the equation

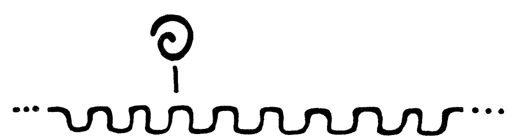
$$\overline{f}| = f.$$

In the form of the liar paradox, the equation $\overline{f}| = f$ becomes $f = -f$, where $-$ means not (or negation). Attempting to evaluate the truth or falsehood leads to an iterant pattern:

$$I: TFTFTFTFTFTFTFTFTFTFTFTFTFTF\ldots$$
$$J: FTFTFTFTFTFTFTFTFTFTFTFTFTFT\ldots$$

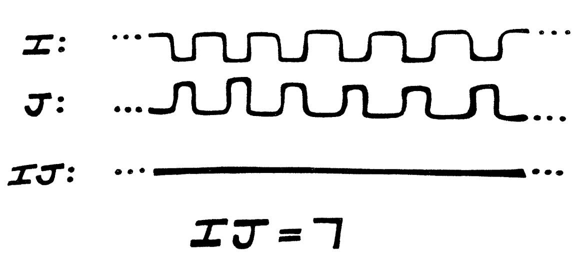
Really two patterns are possible, each obtained from the other by a half-period shift. I have denoted these two patterns above by the letters I and J. I is the pattern of oscillation obtained by initially assuming the truth of a self-falsifying statement. J is the pattern corresponding to an initial assumption of falsehood.

In the more neutral language of Spencer-Brown (1972, 1979), I and J correspond respectively to initial assumptions of markedness or unmarkedness for f in the equation $\overline{f}| = f$. If we think of the solution to $\overline{f}| = f$ as the re-entering mark ⊙, then we see that



any attempt to evaluate the re-entry will set it in an oscillation whose phase is determined by the initial conditions.

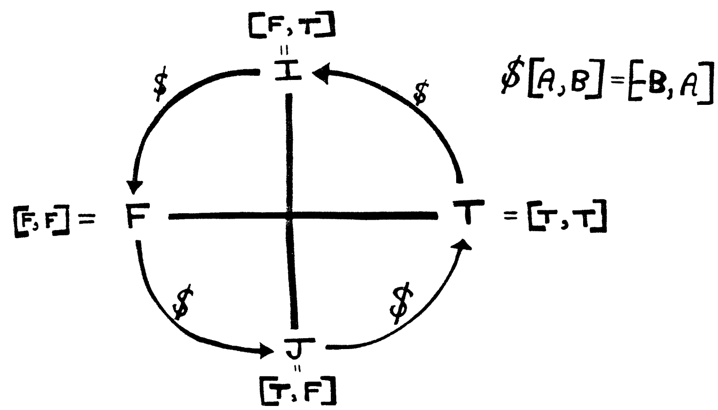
The two solutions I and J correspond formally to a complex number and its conjugate. And they can be combined to create a real value. For example, if I and J are regarded as oscillations between markedness and voidness, then IJ (the simultaneous combination of the oscillations) is always marked and hence represents a marked state.



If we represent I and J by ordered pairs

$$I = [F, T]$$
$$J = [T, F]$$

then we can create a Cartesian cross of real and imaginary logical values as shown below:



Here $T = [T, T]$ and $F = [F, F]$ represent true and false as indicators of a constantly true and consequently false process. The artifice of the ordered pair allows indication of the phase shift between I and J.

In the diagram above, we have also indicated an operation \$ defined by

$$\$[A, B] = [-B, A].$$

Thus,

$$\$T = [-T, T] = [F, T] = I$$
$$\$I = F$$
$$\$F = [T, F] = J$$
$$\$J = T$$

and so we see that this cross of real and imaginary (I, J) Boolean values carries the same properties as the real and imaginary numbers $+1$, -1, $+i$, $-i$. In this formal version, we even have an operator \$ corresponding to the ninety-degree rotation!

In fact, it is very tempting to rewrite it in the form

$$\$\$ = -$$
$$\$(A + \$B) = \$A + \$\$B = \$A - B = -B + \$A$$

and to compare this with

$$\$[A, B] = [-B\ A] \text{ and with}$$
$$i(a + ib) = ia + iib = ia - b = -b + ia.$$

The formalism of the complex numbers is precisely mirrored in this particular unfolding of self-reference.

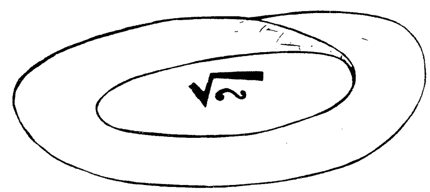
The discussion can be expanded in a number of directions from here. One avenue is the concept of *multiple viewpoint*. Both I and J may be regarded as particular ways of viewing an unending oscillation of T and F. In this sense, they are like two views of a Necker cube illusion, and they represent the way the process of perception splits an apparently existent form into a multiplicity of mutually exclusive and yet related views.

The complex numbers $a + bi$ and the imaginary Boolean values I and J are the simplest mathematical forms that take into account a context combining evaluation and multiplicity.

The imaginary Boolean values become an image of self-reference, a first description of the multiplicity in oneness that is a return to the self.

I believe that it is this resonance that accounts for the unreasonable effectiveness of the complex numbers in mathematics and in physics.

Only the imaginary is real.



Epilogue

This essay has moved from the bare notion of self-reference through many transformations and reformulations of this idea. We have touched on feedback, re-entry, recursion, language, geometry, and topology. None of these excursions is meant to be the final word, but rather the opening for new ideas and greater discussion.

Mathematics is the consequence of what there would be if there could be anything at all.

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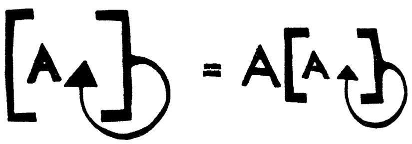
Addendum

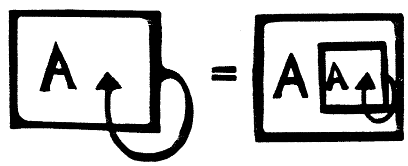
The purpose of this Addendum is to clarify and to extend mathematical issues related to self-reference and re-entry. In the body of the paper, we have used certain notations informally. Some of these are new, and some actually lead to quite deep philosphical and formal questions.

† Title of interest not cited in the text.

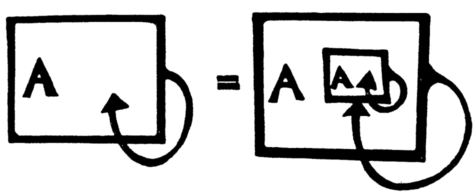
Recall that we have discussed two explicit conventions for re-entry:
First convention: The sign of enclosure is erased.



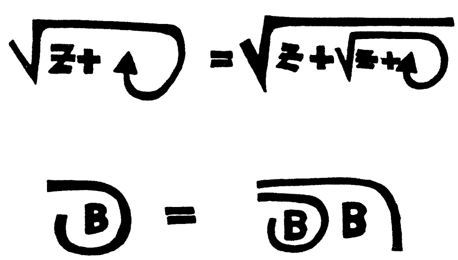
Second convention: The sign of enclosure is not erased.



Essentially, this is a rule for inserting a picture 'in itself'. But we have to note an important point here: the external arrow should be erased if we do not want to obtain the structure:



In using a box with an attached arrow, it is a simple matter to discriminate the arrow, and hence to eliminate it while keeping the box (frame). In other instances, we have followed a similar convention



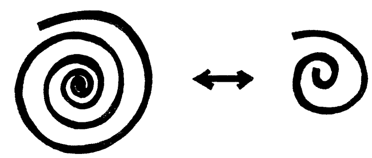
with less explicit instruction as to where to make the cut. (We could have said that the tail of the square root sign begins when the line is no longer horizontal. And a similar convention works for the re-entering mark.)

Obviously, here is a fertile field for graphic and mathematical experimentation. Graphic formalisms hold the possibility for a transition between the subtleties of perceptual experience and the clarity of mathematical formulation. All that is required of such a formalism is that its conventions can be easily explained and repeated.

Another example has been our passing mention of the pictorial formalisms for knots and links.

These formalisms are extraordinarily useful, forming the underpinning of the combinatorial theory of knots. This is a subject that could hardly exist without a diagrammatic language.

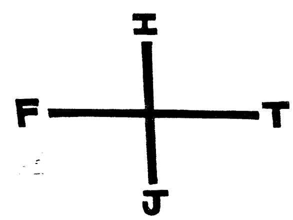
Another direction for investigation involves using re-entry forms to indicate *continuous* operations. A spiral that goes inward infinitely could then be indicated by the first few turns, and we would have the beginnings of a calculus of spirals!



(I am indebted to Vladimir Lefebvre for reminding me of this possibility, and to Frederick Joseph Staley for sharing many such calculi over the past five years.)

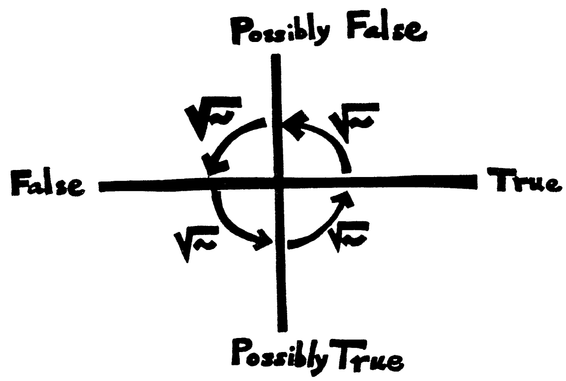
There is an extraordinary potential in the construction of calculi for pattern and re-entry. We encourage the reader to try his/her hand at the creation of this new mathematics!

Finally, I would like to make one further comment about the cross of real and imaginary values in logic:



A useful interpretation ensues if we consider the vertical axis to be an axis of *possibility*, while the horizontal axis represents *necessity*. Thus 'true' and 'false' are states in the domain of necessity, while 'possibly true' and possibly false' are states in the domain of 'possibility'. As anyone intent upon the solution of a difficulty is actually aware, there is an enormous difference between attitudes of possible truth and possible falsehood.

The 'square root of negation' is the operator of fourfold rotation:



In this mode, the square root of negation is the sensible precursor to the square root of minus one.

Only by rotating into the realm of possibility do we enter the domain of the self.