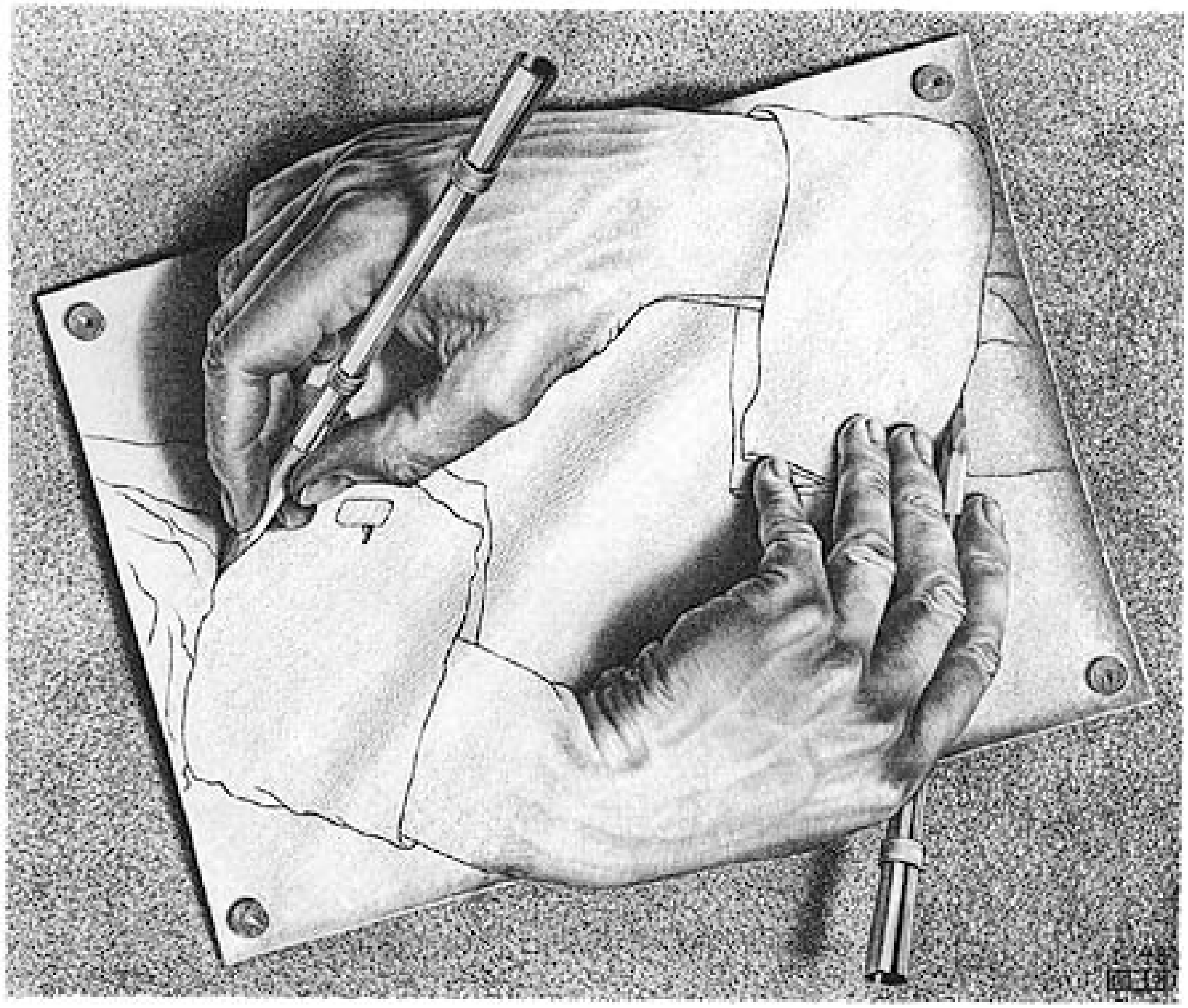
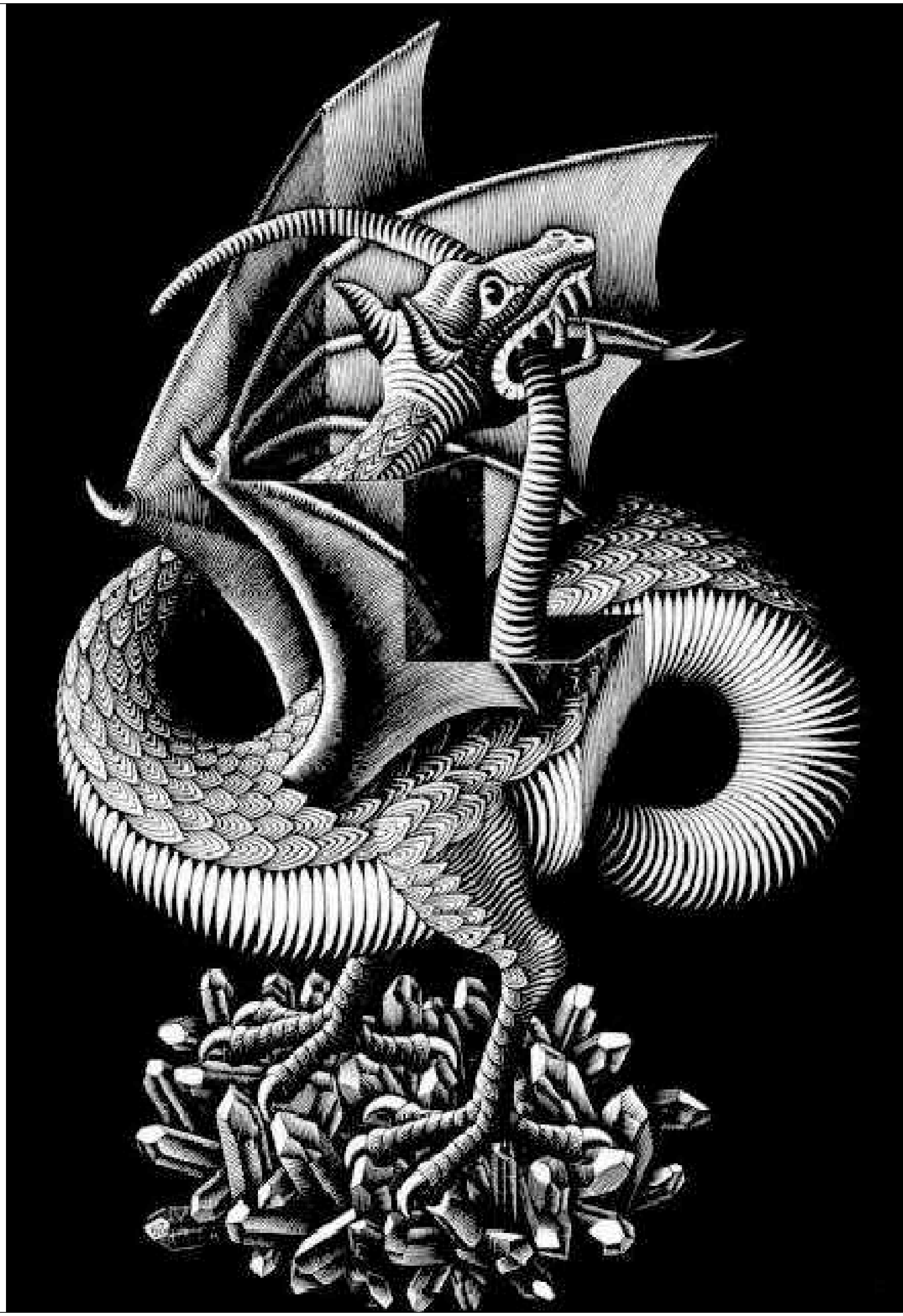
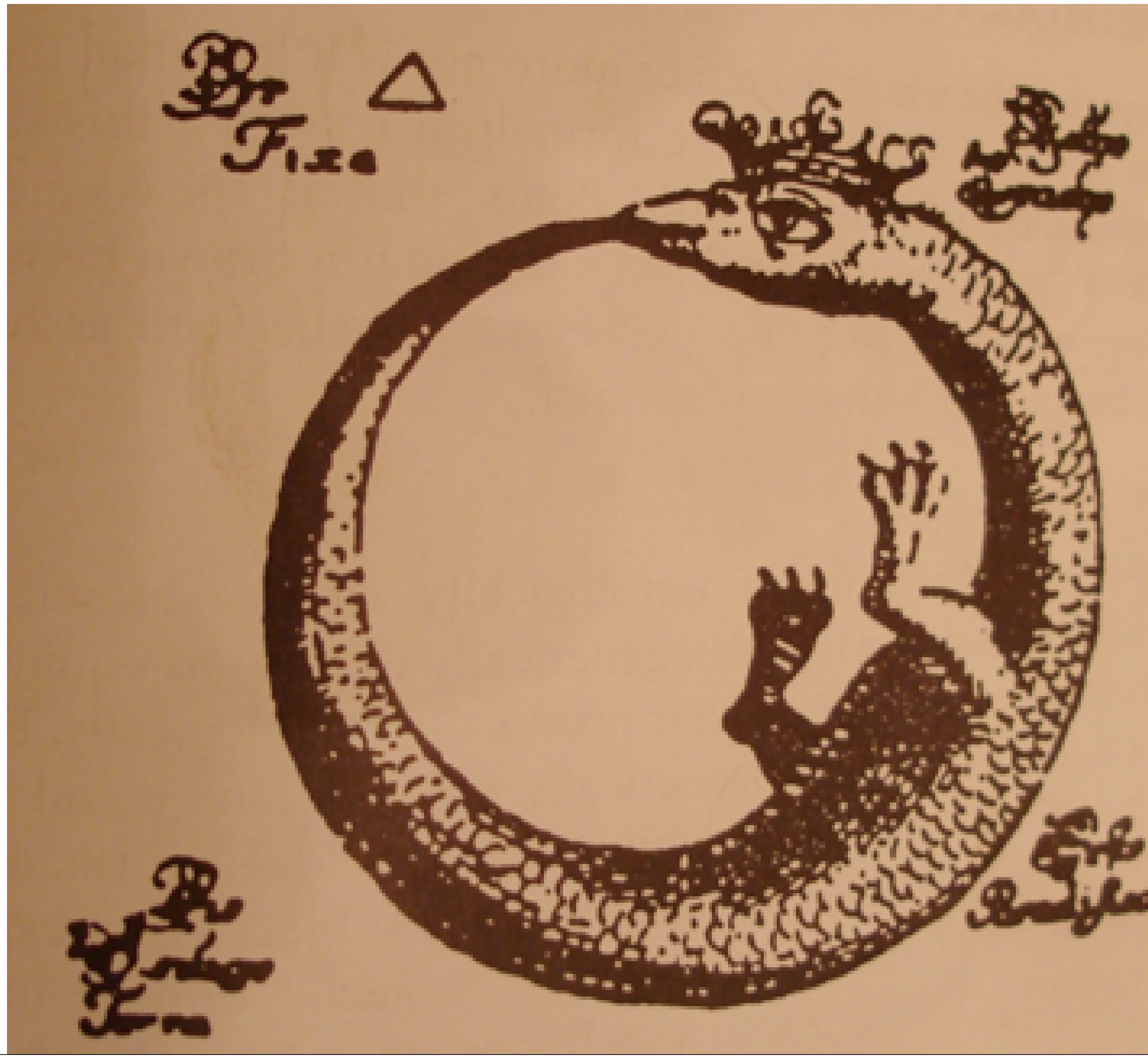
Reflexivity

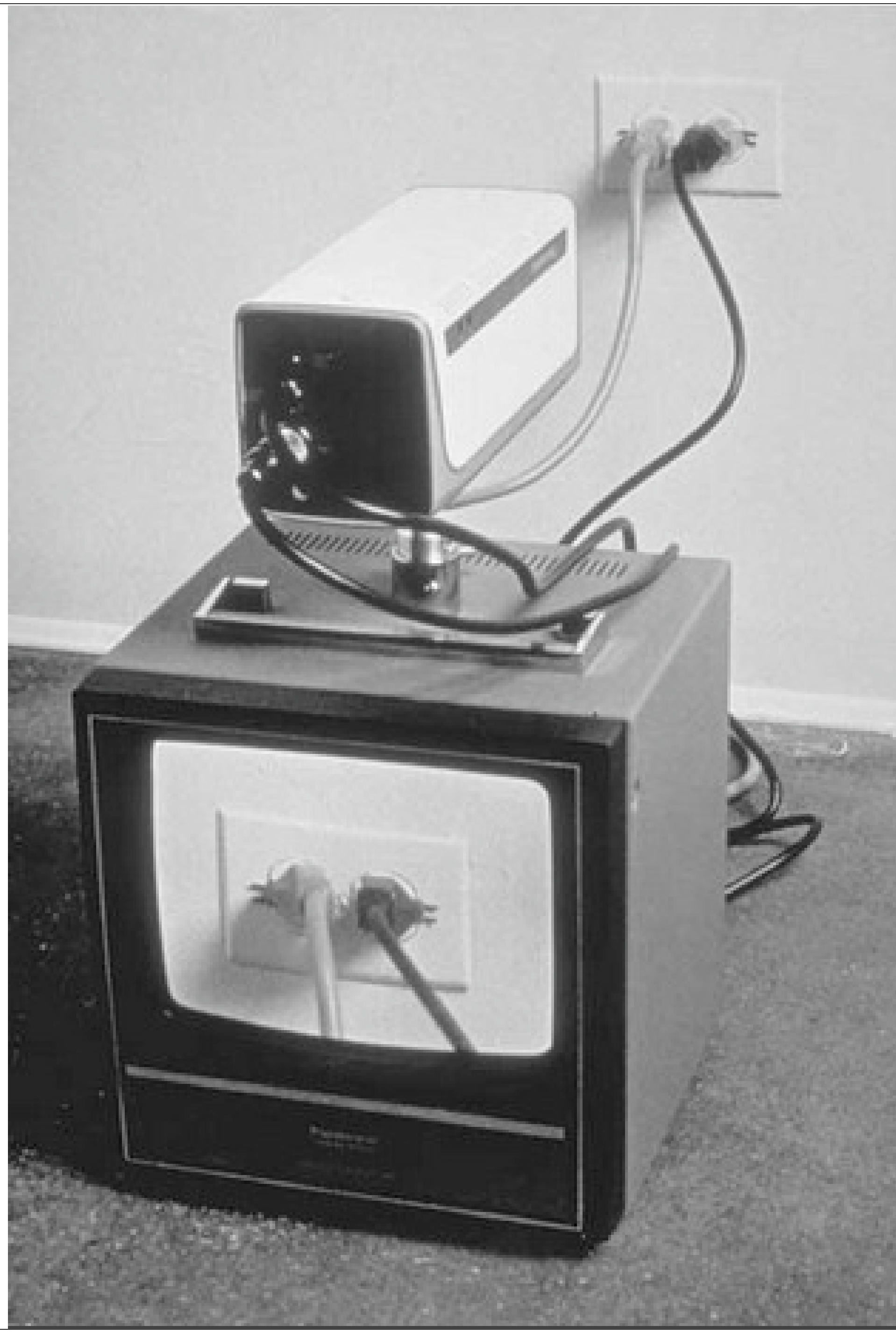
Louis H. Kauffman, UIC

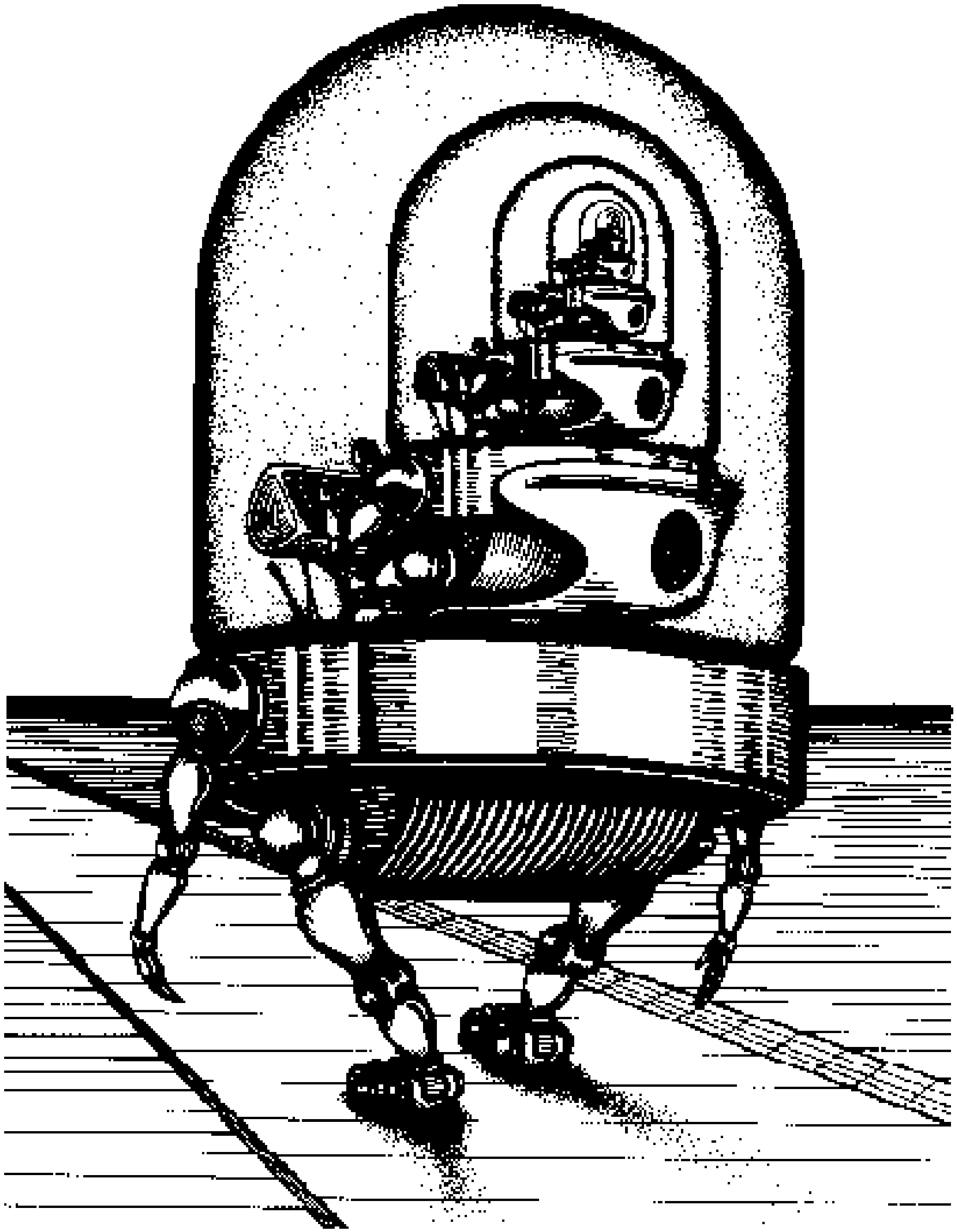




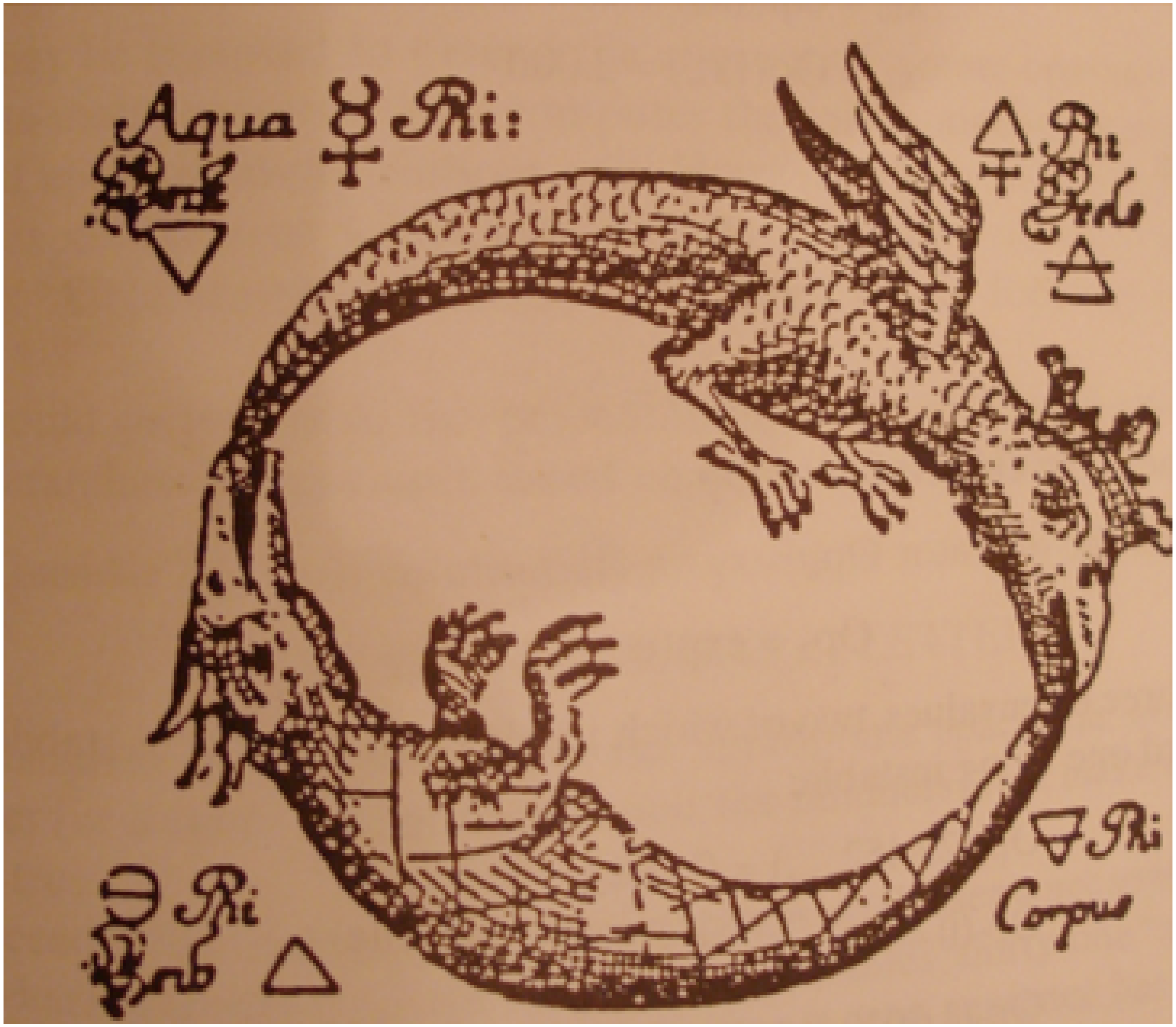
Reflexivity refers to a relationship between an entity and itself.







Reflexivity refers to mutuality of relationship as well.



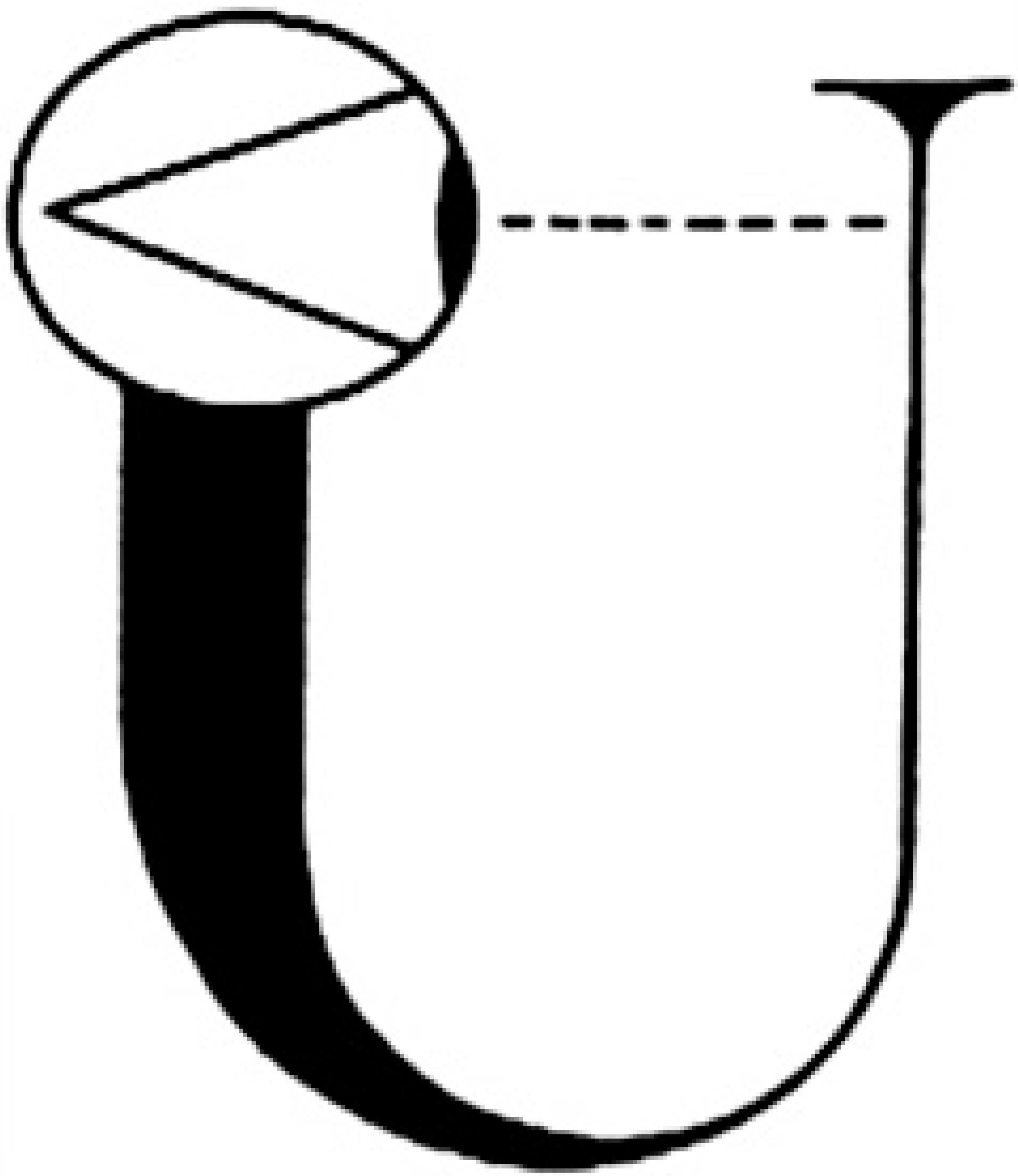
Simplicity

A logician saves the life of a tiny space alien. The alien is very grateful and, since she's omniscient, offers the following reward: she offers to answer any question the logician might pose. Without too much thought (after all, he's a logician), he asks: "What is the best question to ask and what is the correct answer to that question?" The tiny alien pauses. Finally she replies, "The best question is the one you just asked; and the correct answer is the one I gave."

Indirect Self-Reference

Gebstadter, Egbert B. Copper, Silver, Gold: an Indestructible Metallic Alloy. Perth: Acidic Books, 1979. (Two-hundred-fortieth-luniversary edition, Perth: Acidic Books, 1999.)

A formidable hodge-podge, turgid and confused — yet remarkably similar to Douglas Hofstadter's first work, and appearing in its well-annotated bibliography. Professor Gebstadter's Shandean digressions include some excellent examples of indirect self-reference. Of particular interest is a reference in its own well-annotated bibliography to an isomorphic, but imaginary, book.




Ceci n'est pas une pipe.
Magritte


Ceci n'est pas une pipe

One can be aware of
one’s own
thoughts.

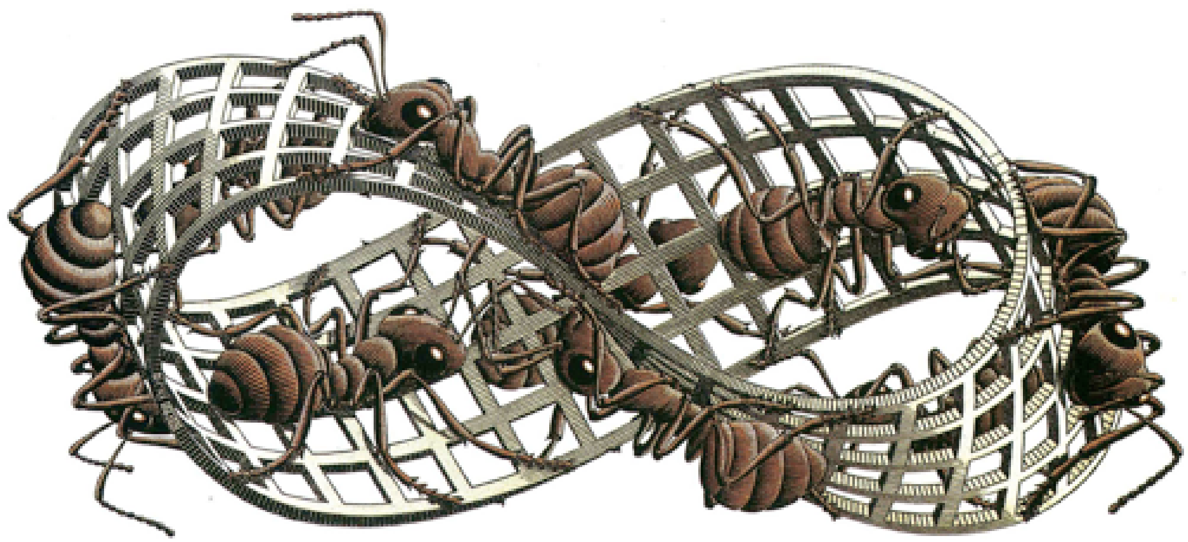
An organism produces
itself
through its
own productions.

A market is composed of
individuals
whose actions
influence the market
just as the
actions of the
market influence
these
individuals.

The participant is
an observer
but
not
an
objective observer.

There is no objective observer.

There is no objective observer, and yet
objects, repeatablity,
a whole world of actions,
and a reality to be explored
arise
in the relexive domain.



The object is both an element of a world
and
a symbol for the process of its
production/observation.

An object, in itself, is a symbolic
entity,
participating in a network of interactions,
taking on its apparent solidity and stabilty
from these interactions.

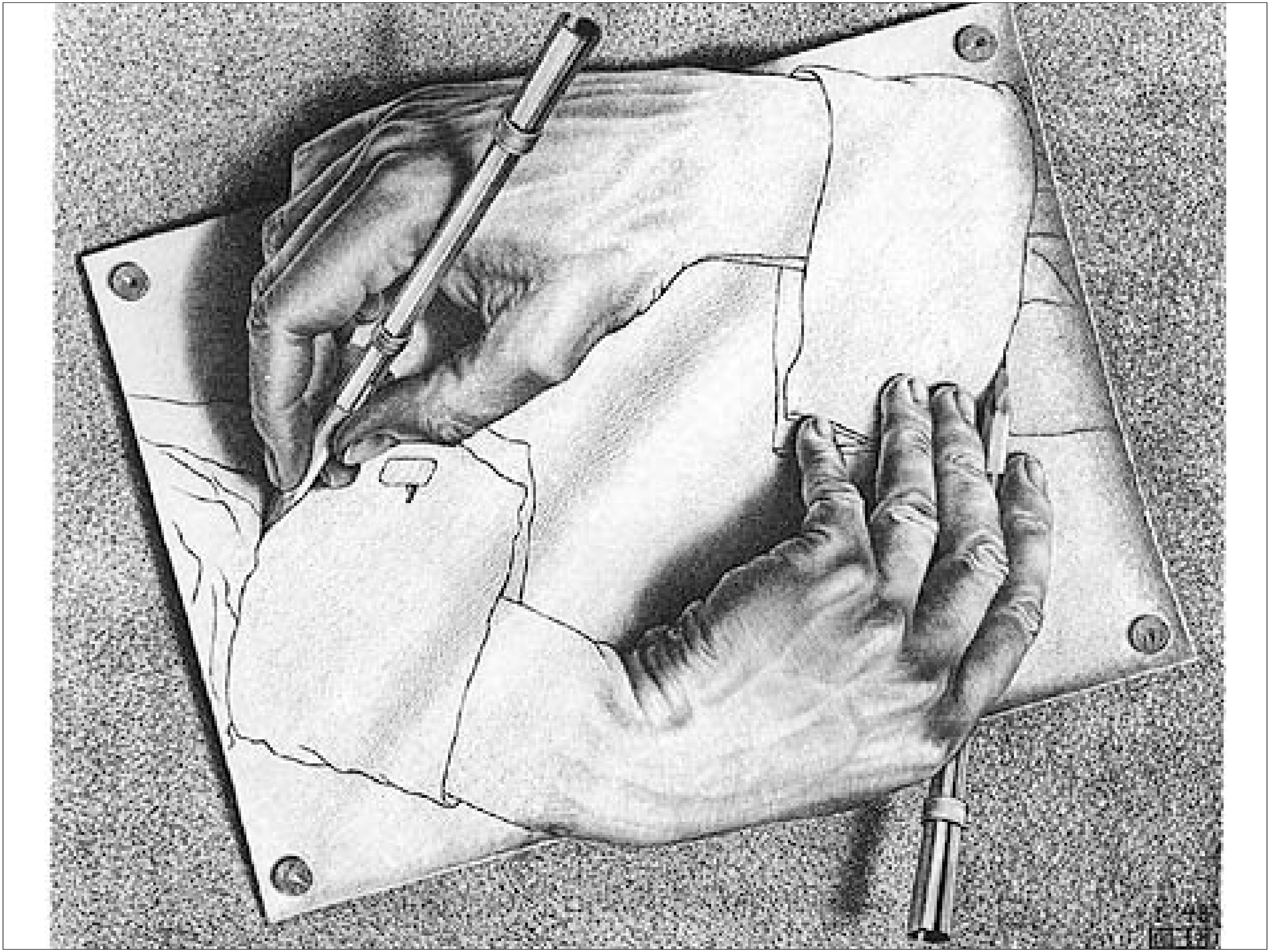
We ourselves are such objects,
we as human beings are
“signs for ourselves”
a concept
originally due to the
American philosopher
C.S. Peirce.

In an observing system, what is observed is not distinct from the system itself, nor can one make a complete separation between the observer and the observed. The observer and the observed stand together in a coalescence of perception. From the stance of the observing system all objects are non-local, depending upon the presence of the system as a whole. It is within that paradigm that these models begin to live, act and enter into conversation with us.

The ground of discussion is
not fixed beforehand.

The space grows in the hands of those
who explore it.

Infinity beckons as an indicator
of process.



Referential and Recursive Domains

We would like to define the concept of a reflexive domain.

The very act of making definitions is itself reflexive.

So any definition that we make will not be all that is possible, and it may even miss the key point!

Nevertheless, we shall try,
keeping in mind that
any formalization is
really an example and not the whole.

There is freedom in this attititude.
You do not have to produce the
Theory of Everything
if
Everything is
Reflected in each
Theory.

Reflexive Domain

A reflexive domain D is a space where
every object is a transformation,
and every transformation
corresponds uniquely to an object.

$$D \longleftrightarrow [D,D]$$

In a reflexive domain
Actions and
Objects
are
Identical.

Eigenforms Exist in Reflexive Domains

Let D be a reflexive domain.

Theorem.
Every transformation T of a reflexive domain has a fixed point.

Proof. Define a new transformation G by

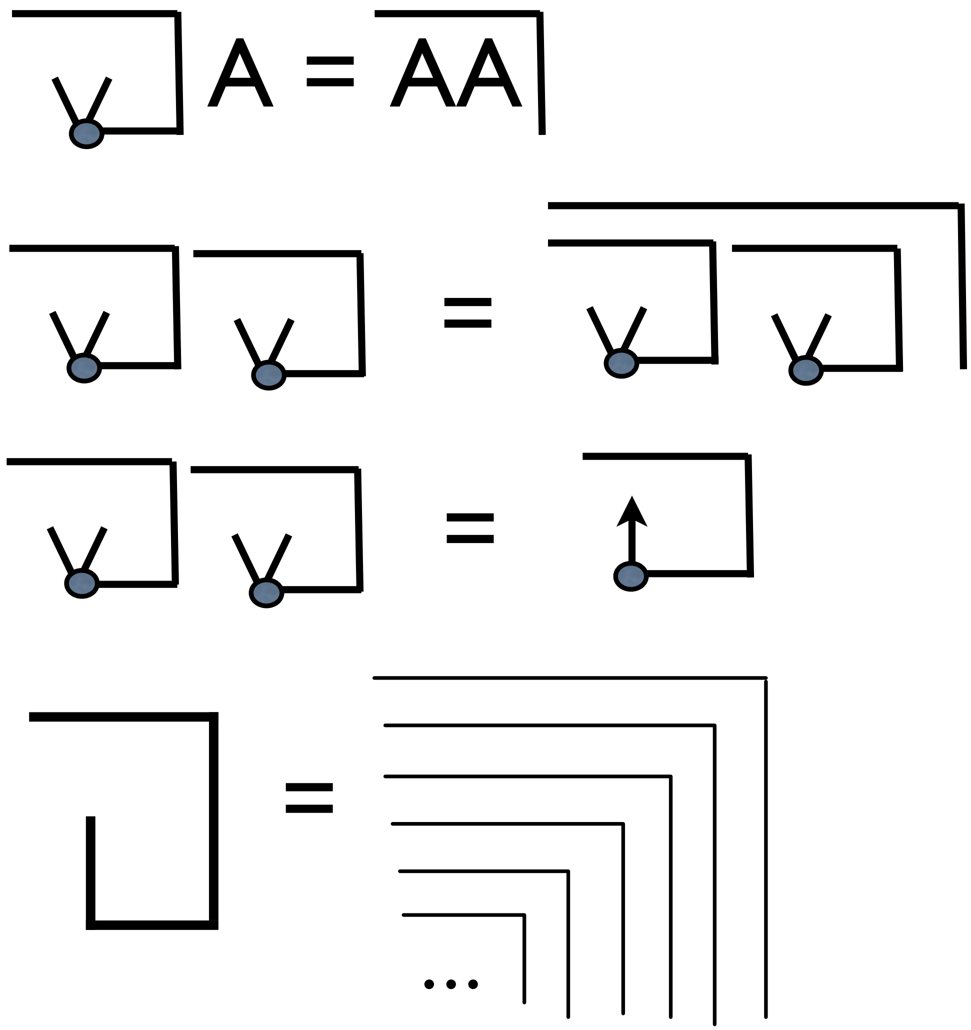
$$Gx = T(xx).$$

Then $GG = T(GG)$.

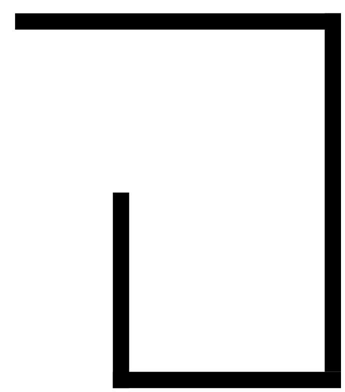
QED.

$$Gx = T(xx) \quad \Rightarrow \quad GG = T(GG)$$

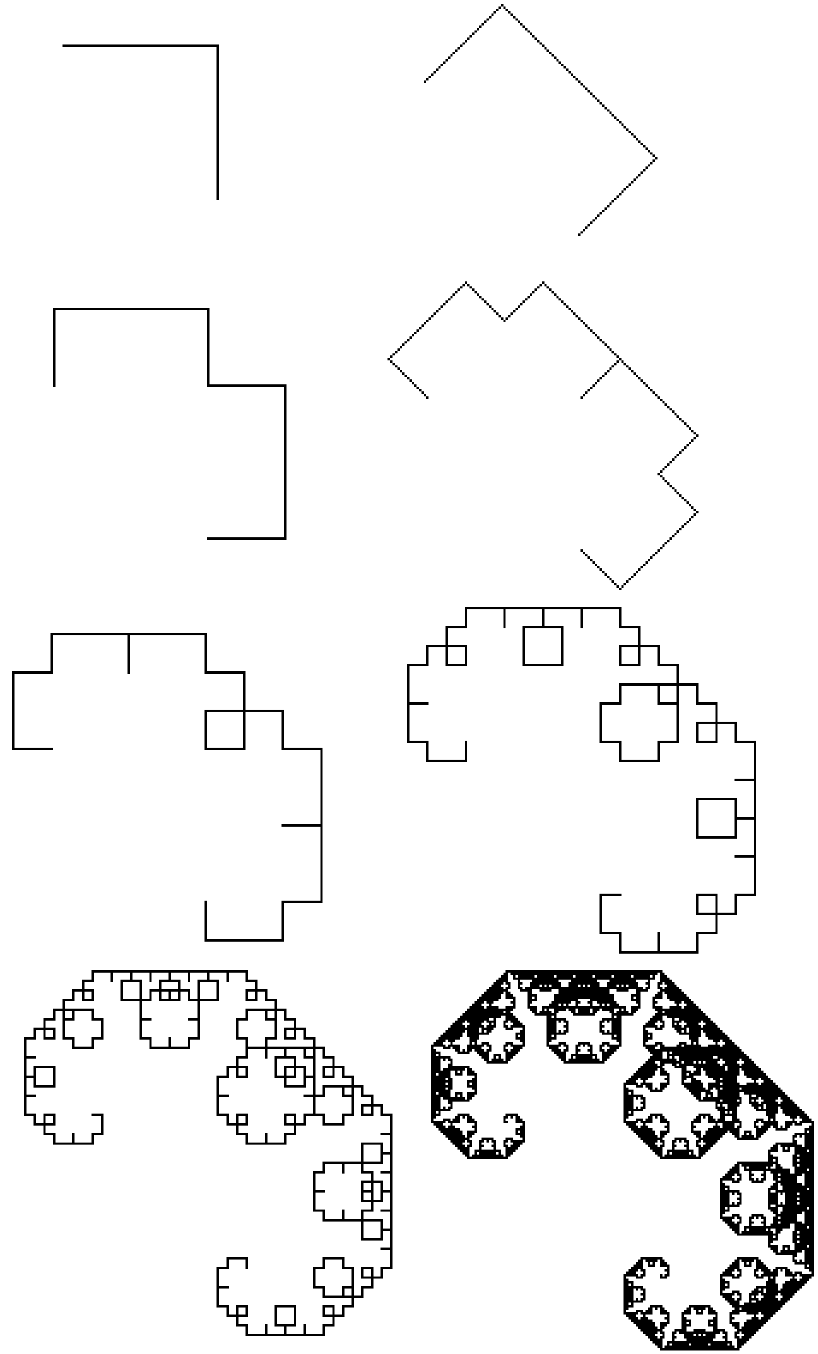
The Duplicating Gremlin Creates The Re-entering Mark.

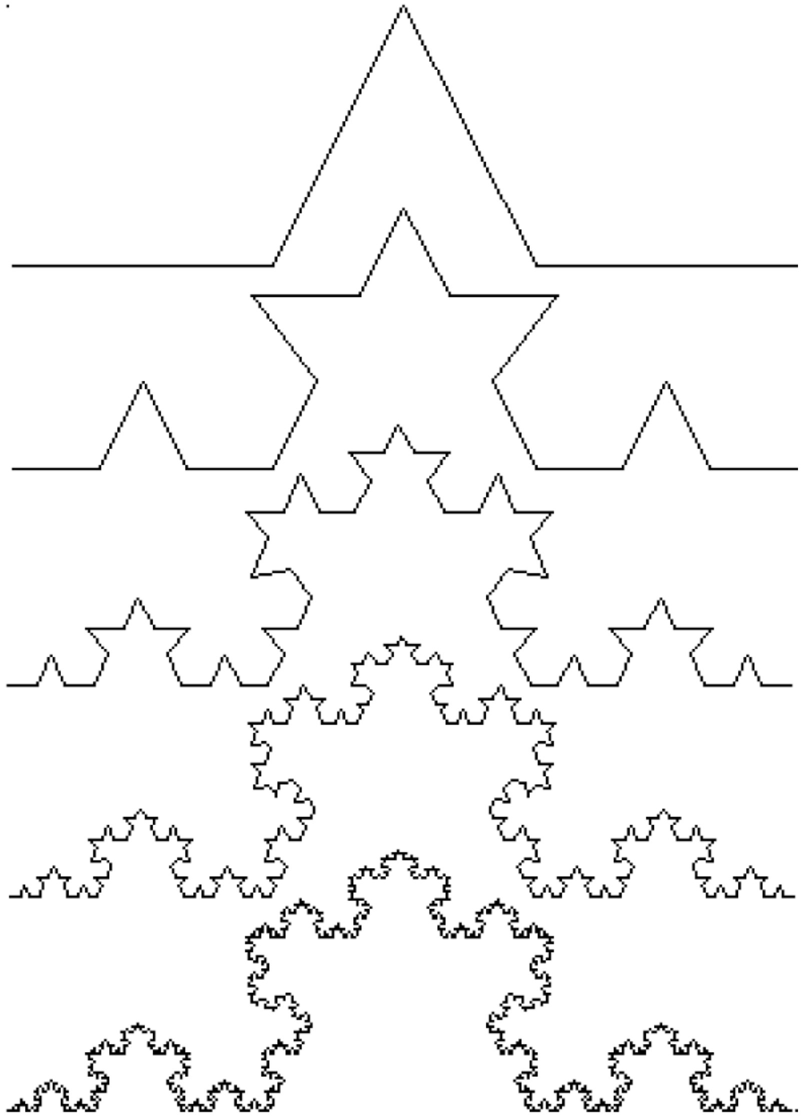


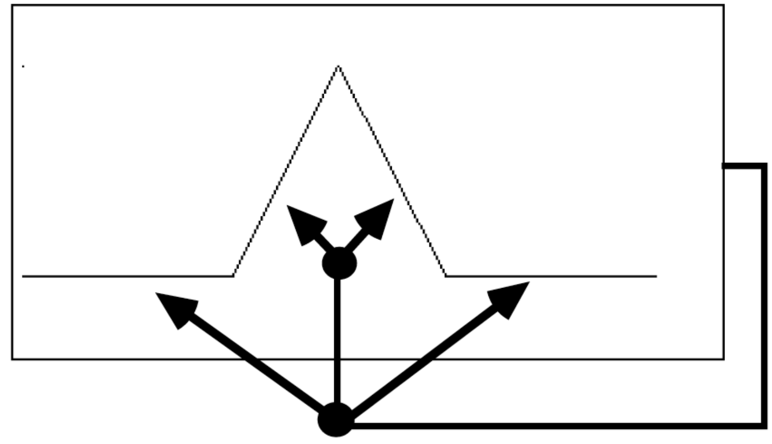
A Form Re-enters its Own Indicational Space.



Fractal Re-entering Mark







$$\mathbf{K} = \mathbf{K}\{\mathbf{K}\ \ \mathbf{K}\}\mathbf{K}$$

The Framing of
Imaginary Space.

Describing Describing

Describing Describing

Consider the consequences of
describing
and then
describing that description.

We begin with one entity:

*

And the language of the numbers:
1,2,3.

Yes, just ONE,TWO,THREE.

*

Description: “One star.”

1*

Description: “One one, one star.”

111*

Description: “Three ones, one star.”

311*

Description: “One three, two ones, one star.”

13211*

Describing Describing

*

1*

111*

311*

13211*

111312211*

311311222111*

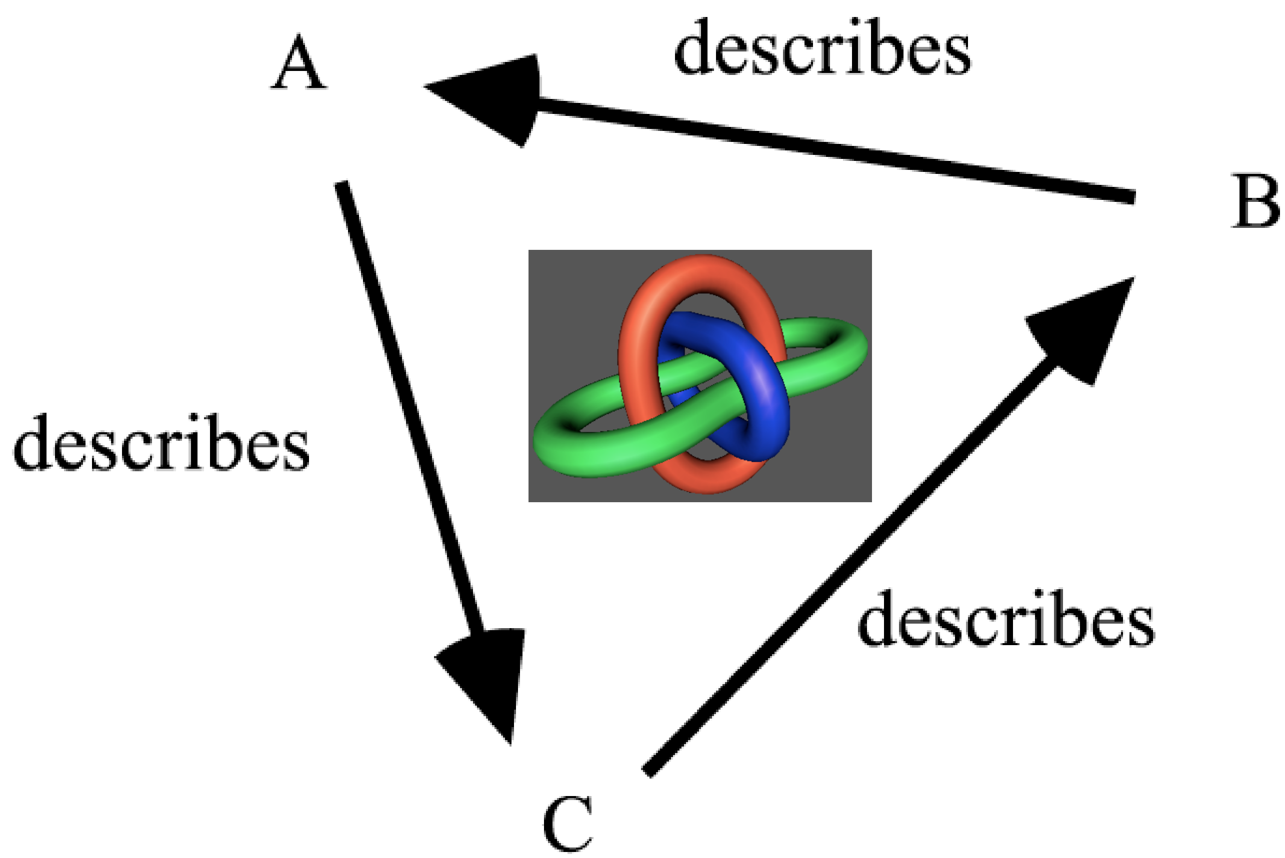
1321132132311*

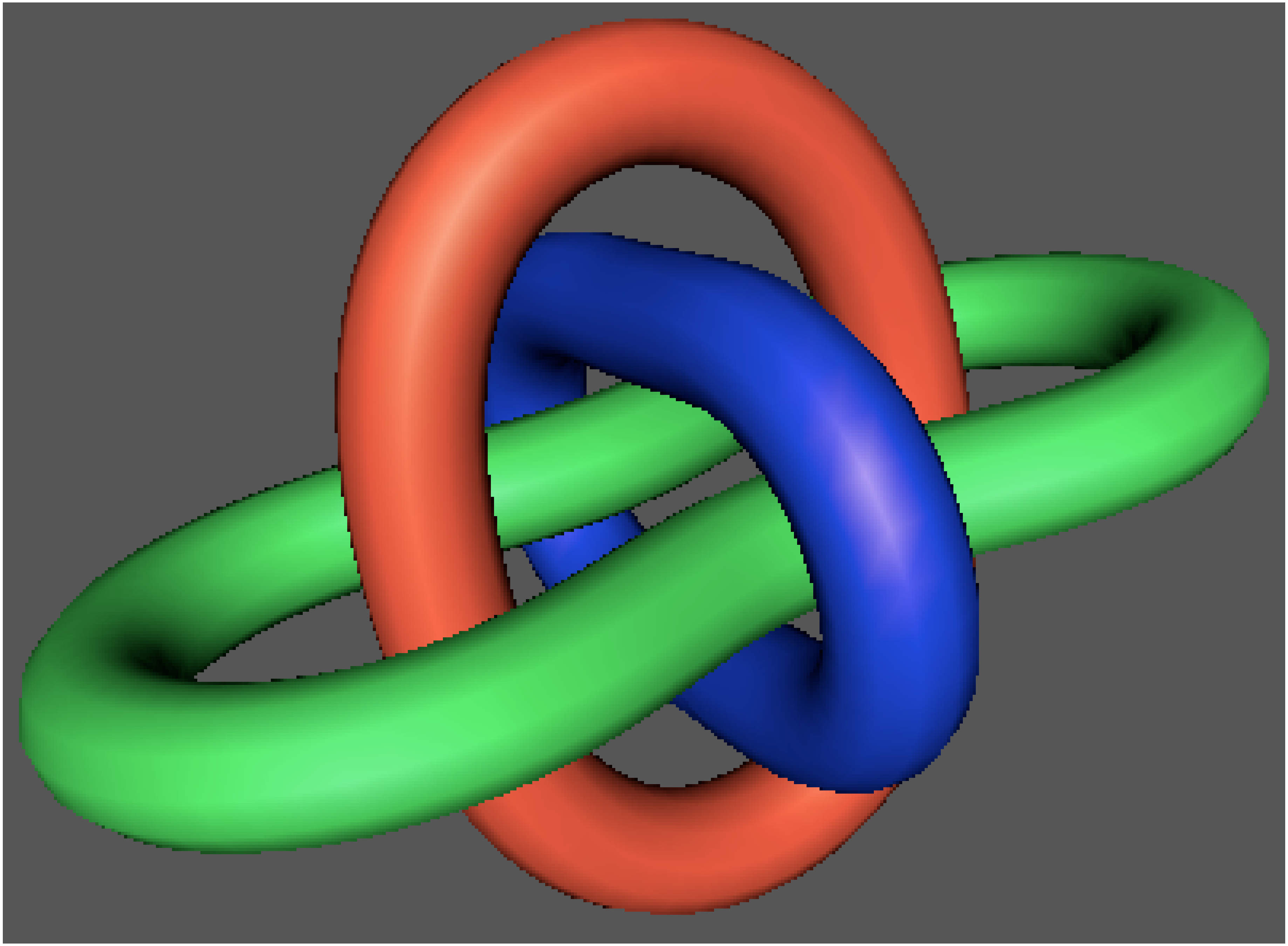
11131221131211131213211*

$$\mathbf{A = 11131221131211132221\ldots}$$

$$\mathbf{B = 3113112221131112311332\ldots}$$

$$\mathbf{C = 132113213221133112132123\ldots}$$





The Form
We take to exist
Arises
From
Framing
Nothing.

G. Spencer-Brown

Eigenforms can
transcend the domains
in which they
originate.

An Example

$$T(x) = 1 + ax$$

$$T(T(x)) = 1 + a(1 + ax) = 1 + a + aax$$

$$E = 1 + a + aa + aaa + aaaa + \ldots$$

$$E = 1 + a(1 + a + aa + aaa + \ldots) = 1 + aE$$

$$E = 1 + aE$$

$$E = T(E).$$

What about $a = 2$?

$$E = 1 + 2 + 4 + 8 + \ldots$$

$$E = 1 + 2E$$

implies that

$$E = -1.$$

$$-1 = 1 + 2 + 4 + 8 + \ldots \quad !!?$$

The meaning is hidden:

$$1+2 = -1 + 4$$
$$1+2+4 = -1 + 8$$
$$1+2+4+8 = -1 + 16$$
$$\ldots$$

1 + 2 + 4 + 8 + ... = "-1 + 2^{Infinity}"

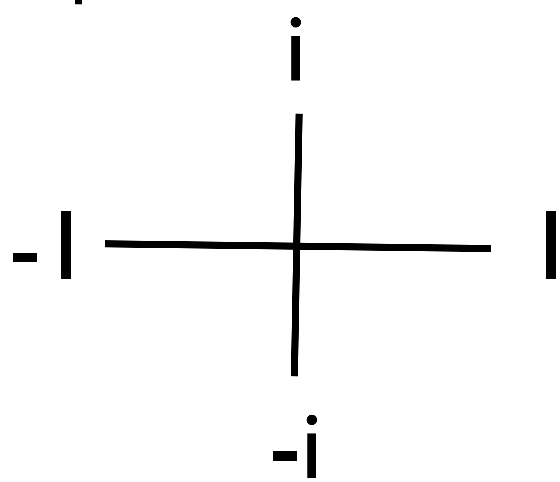
The eigenform always exists, but it may be imaginary with respect to our present Reality.

If $i = -1/i$,

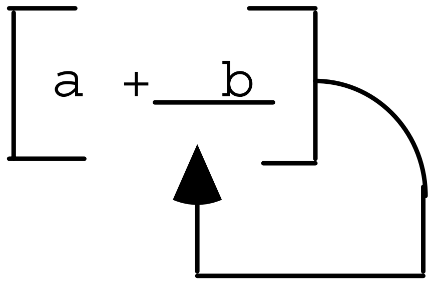
then

$i\,i = -1$.

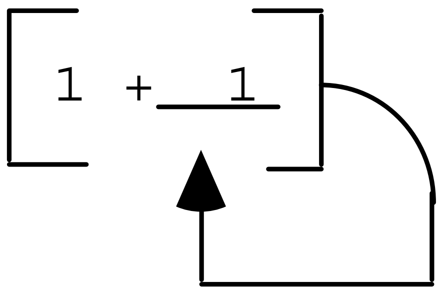
There is no real number whose square is minus one.



$f(x) = a + b/x$

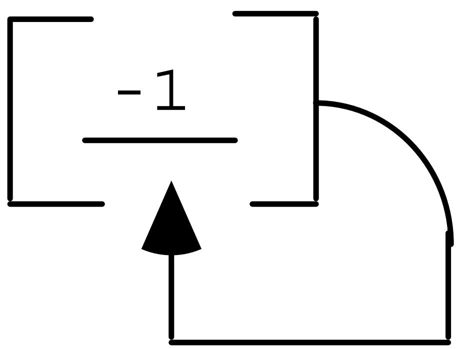
F = 

$f(F) = a + b/F = F$



$$= \frac{1 + \sqrt{5}}{2}.$$

Irrational



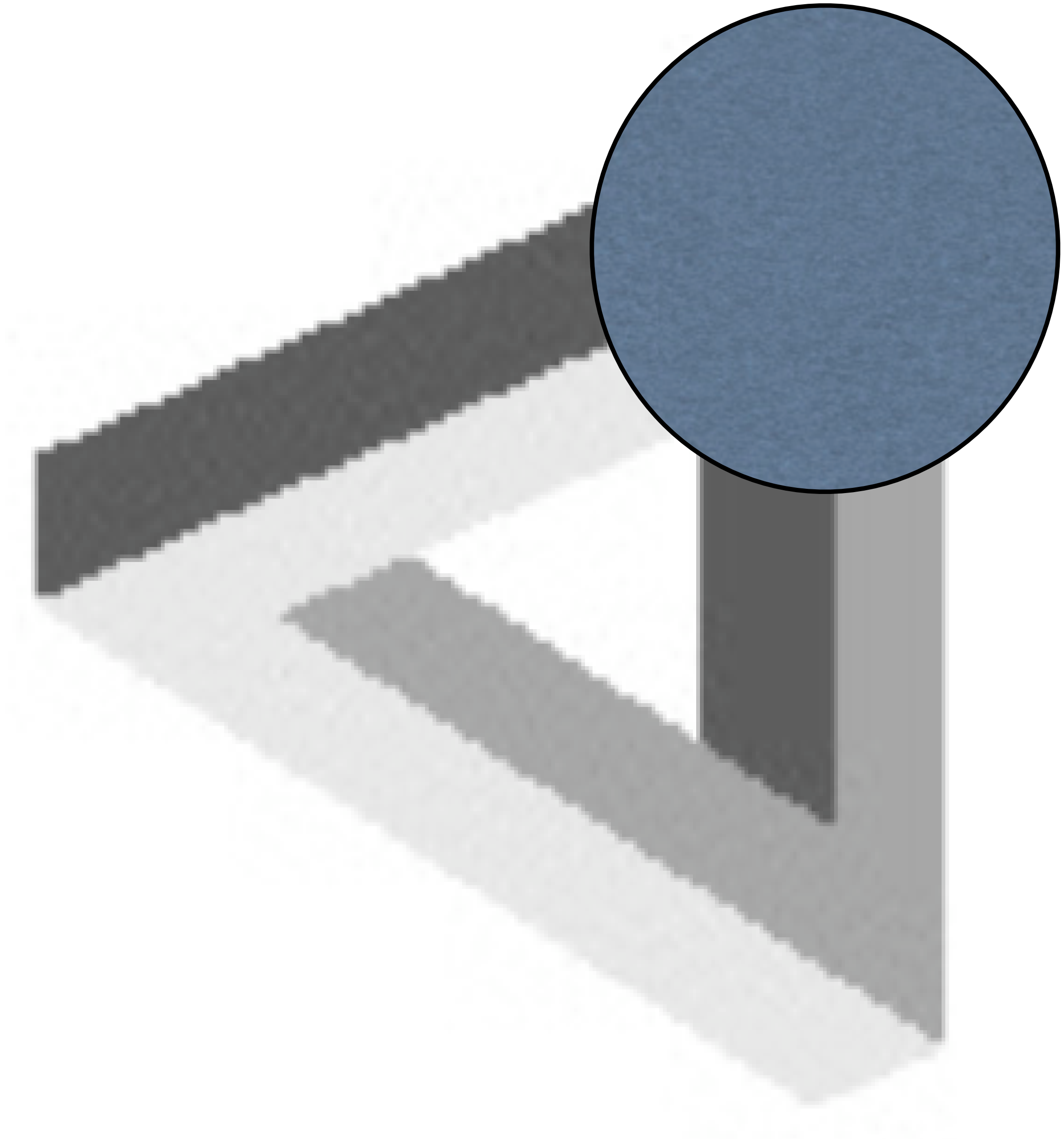
$= i$

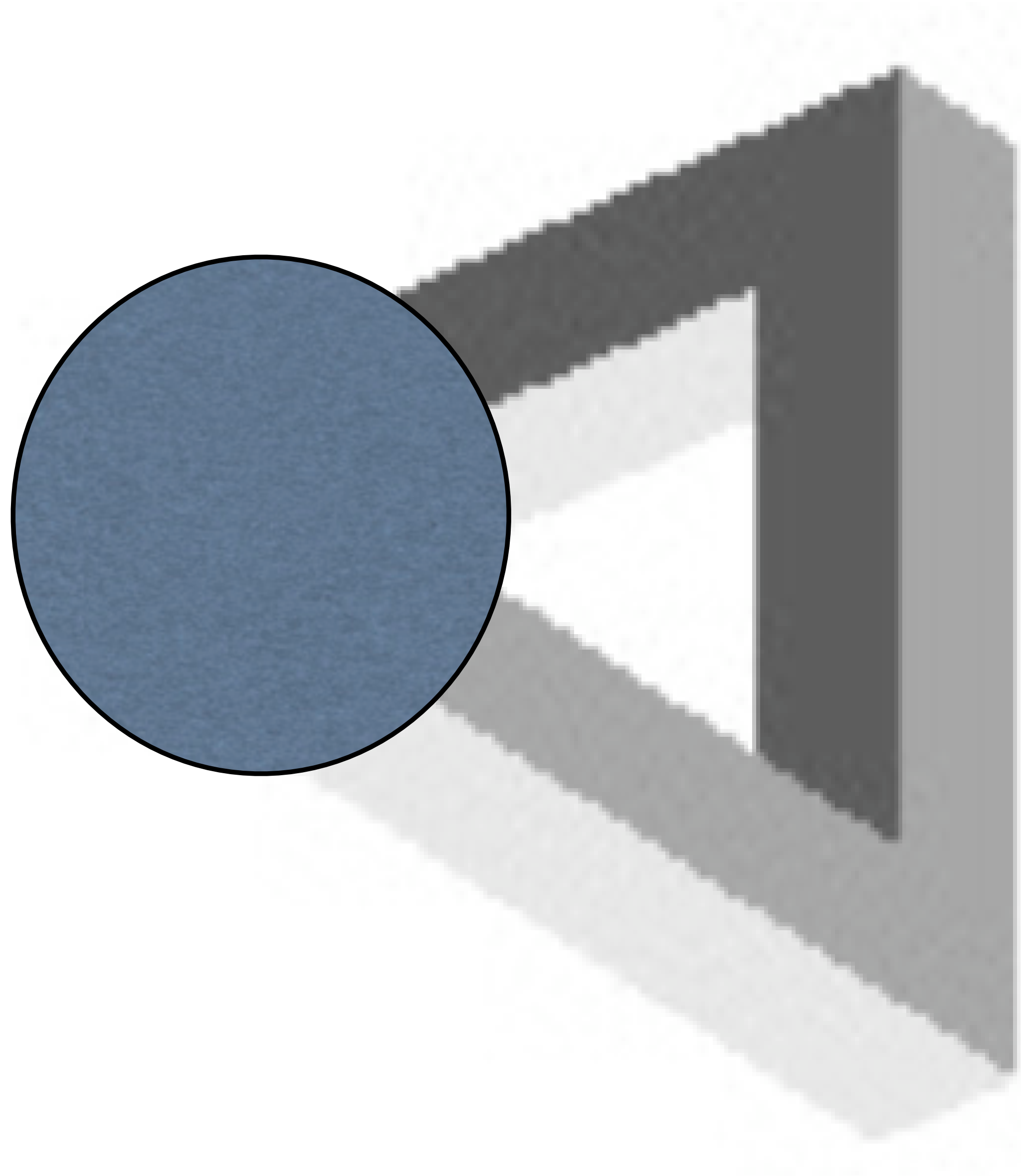
Imaginary

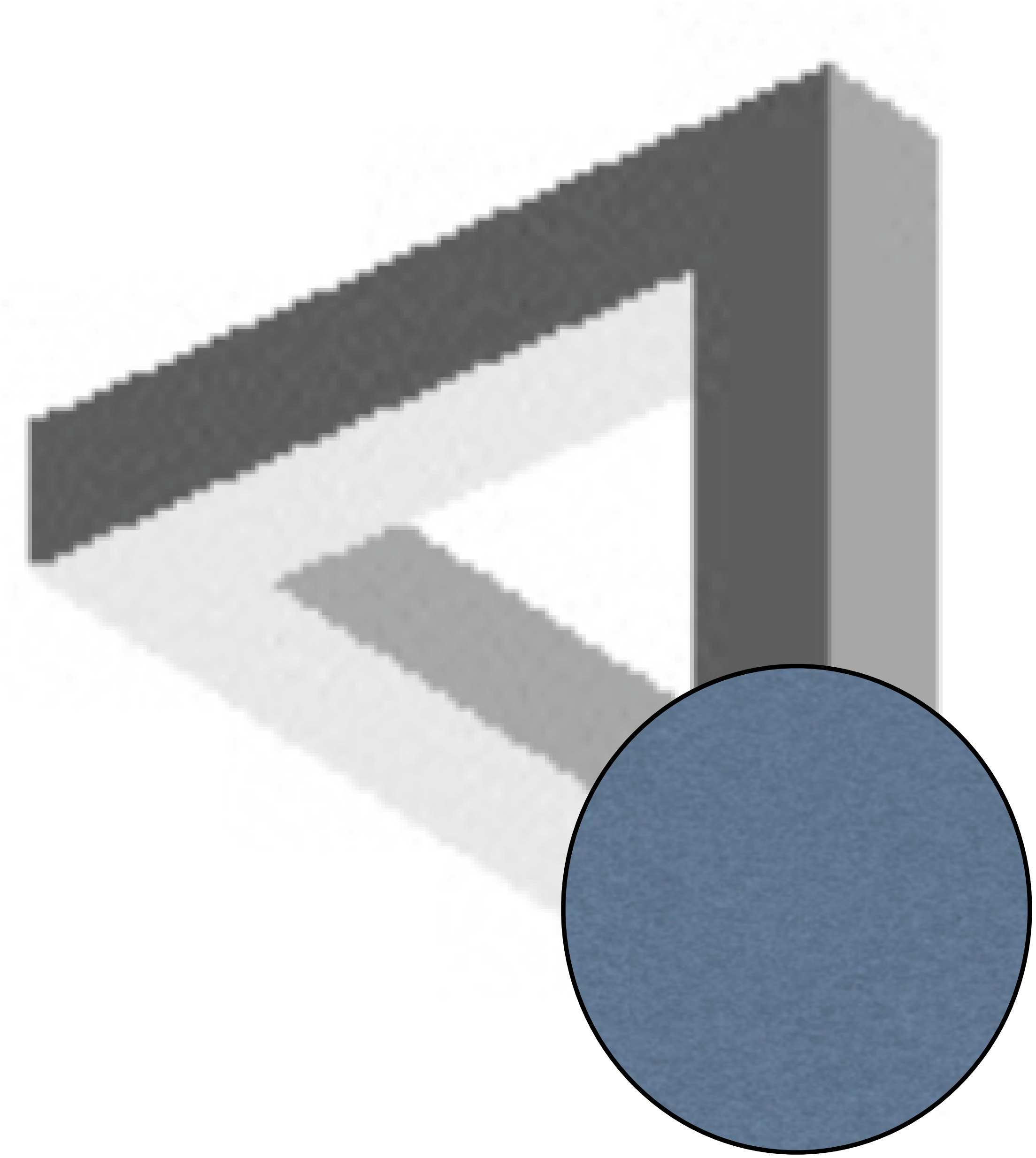
$\updownarrow$

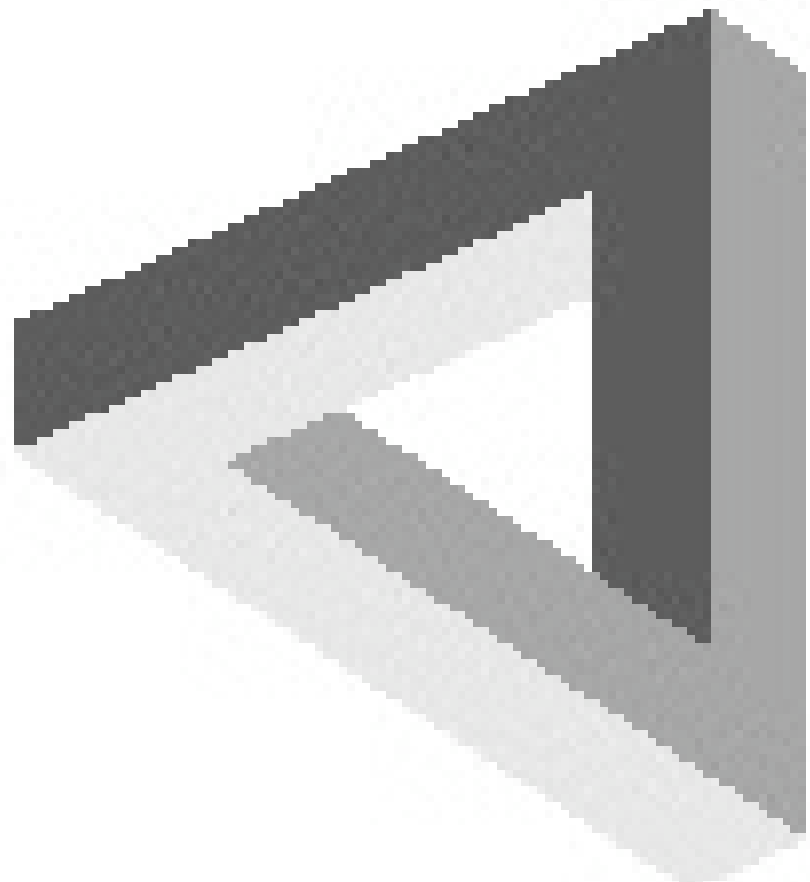
... +1 -1 +1 -1 +1 -1 ... Iterant

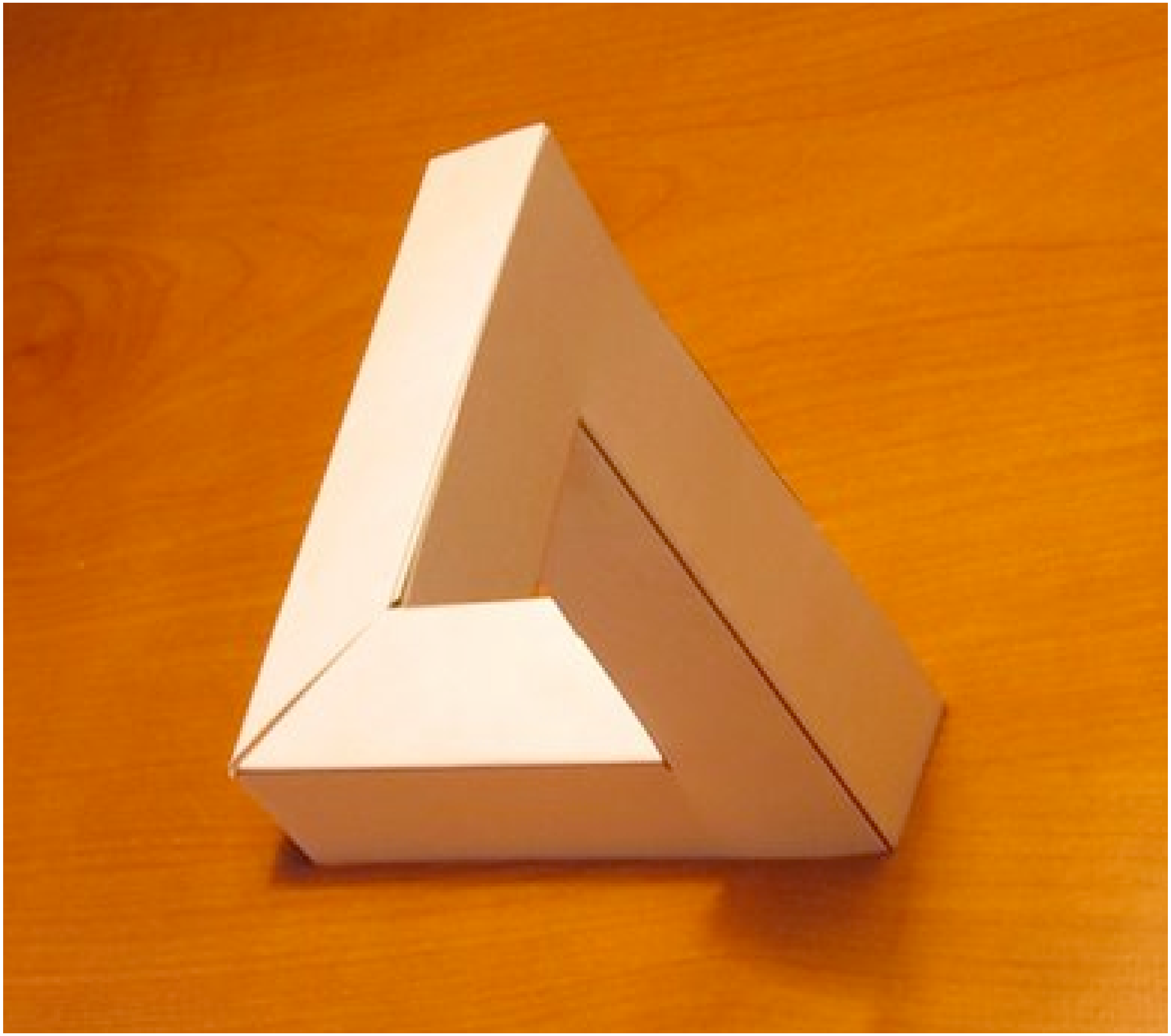
The Non-Locality of Impossibility



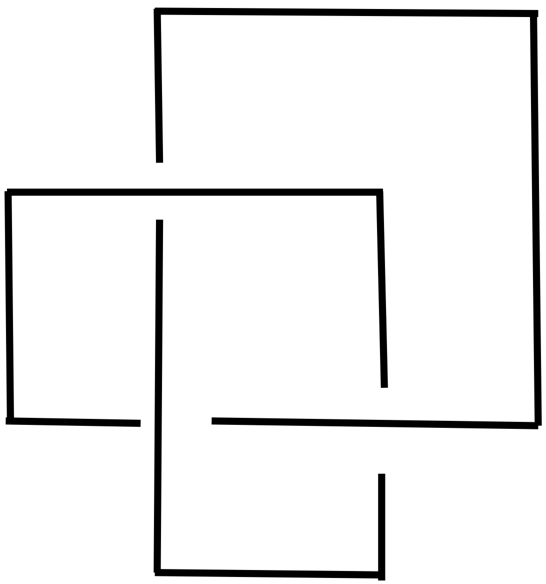


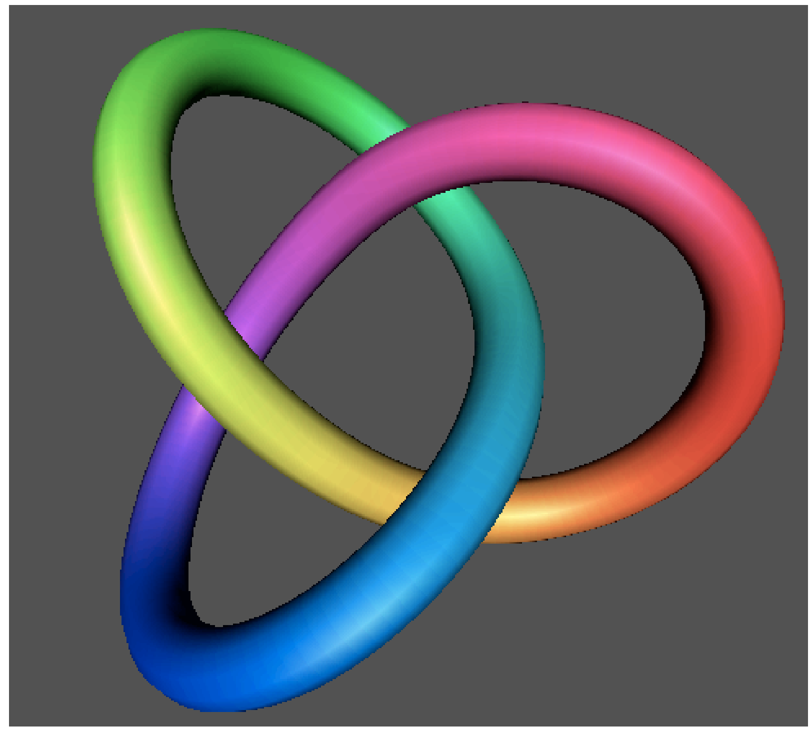






The Imaginary and The Real





Set Theory

A set is a collection of objects.
These objects are the members of the set.
Two sets are equal exactly when
they have the same members.

The simplest set is the empty set
{ }.

Cantor's Theorem in a Nutshell: P(X) > X.

Let AB mean that *B is a member of A.*

Cantor's Theorem. Let S be any set (S can be finite or infinite). Let P(S) be the set of subsets of S. Then P(S) is bigger than S in the sense that for any mapping F: S -----> P(S) there will be subsets C of S (hence elements of F(S)) that are not of the form F(a) for any a in S. In short ,the power set P(S) of any set S is larger than S.

Proof. Suppose that you were given a way to associate to each element x of a set S a subset F(x) of S. Then we can ask whether x is a member of F(x). Either it is or it isn't. So lets form the set of all x such that x is not a member of F(x). Call this new set C. We have the defining equation for C :

$$Cx = \sim F(x)x.$$

Is C =F(a) for some a in S?
If C=F(a) then for all x we have
F(a)x = ~F(x)x.
Take x =a. Then
F(a)a = ~F(a)a.
This says that a is a member of F(a) if and only if a is not a member of F(a). This shows that indeed C cannot be of the form F(a), and we have proved Cantor's Theorem that the set of subsets of a set is always larger than the set itself. //

Cantor's Paradise is Not a Member of Itself.

Let Aleph denote all sets whose members are sets.

Think of Aleph as all sets generated from the empty set
by possibly infinite processes.

Suppose that Aleph itself is a set.

Note that every object in Aleph is a set of sets.
Hence every object in Aleph is a subset of Aleph.
And by the same token
(take note of this figure of speech!)
every subset of Aleph is a collection of sets,
and hence is a member of Aleph.

Therefore P(Aleph) = Aleph.

Therefore Aleph is not a set!!

Russell's Paradox

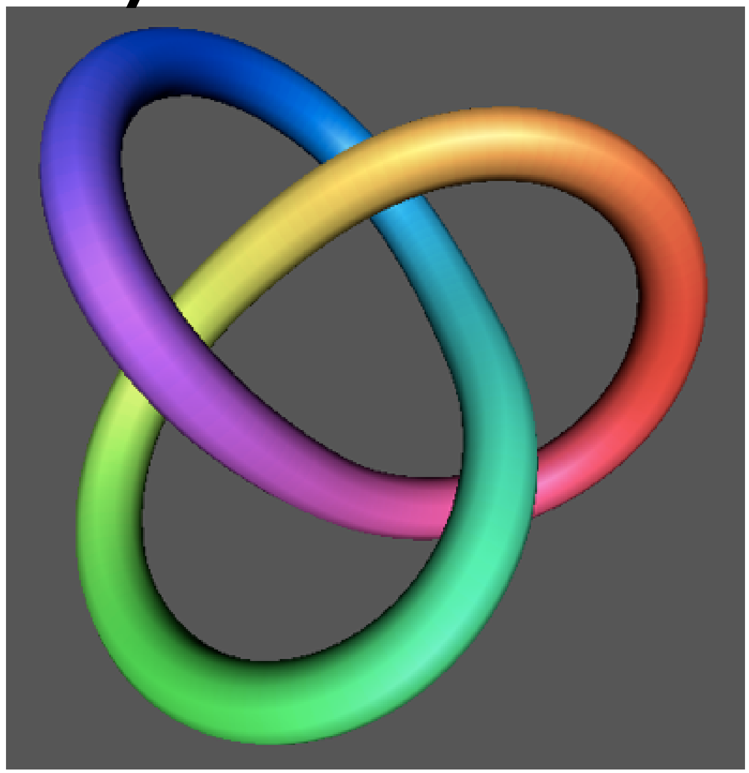
Rx = ~xx

RR = ~RR

R is the set of all sets that are not members of themselves.

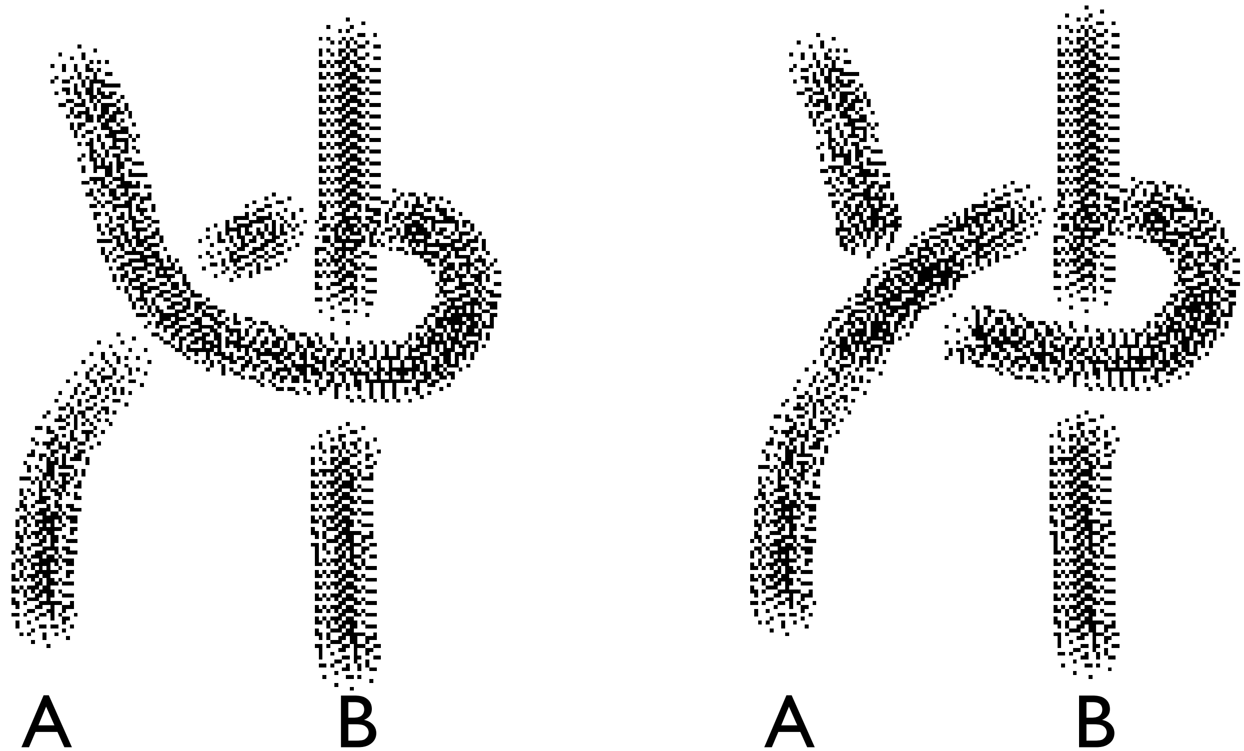
R is a member of itself
if and only if
R is not a member of itself.

Self-Mutuality and Fundamental Triplicity



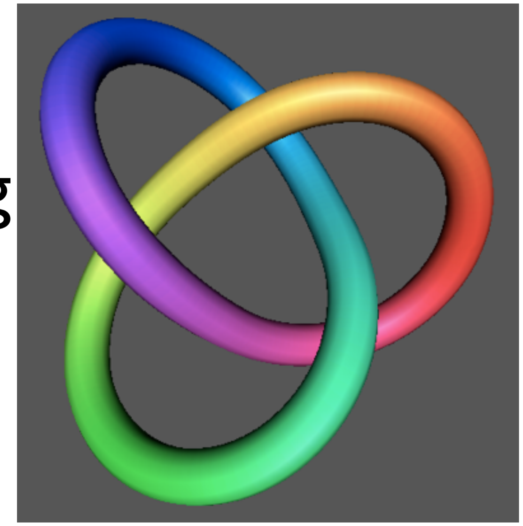
Trefoil as self-mutuality.
Loops about itself.
Creates three loopings
In the course of
Closure.

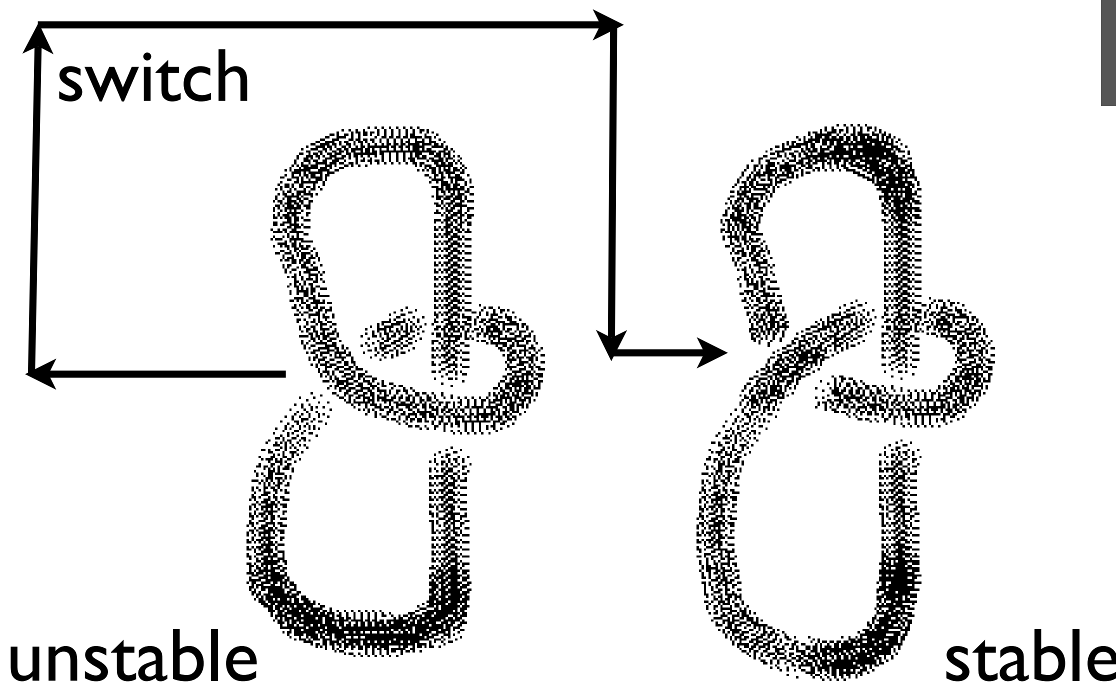
Observation as Linking



A observes B

Self-Observation and Observing Observing

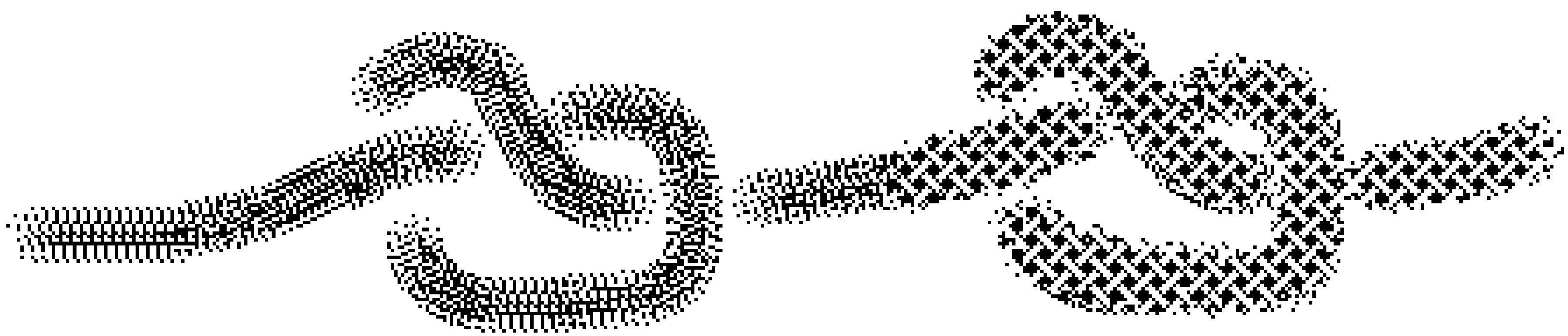




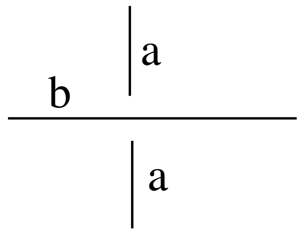
A observing A

Patterned Integrity

The knot is information independent
of the substrate that carries it.

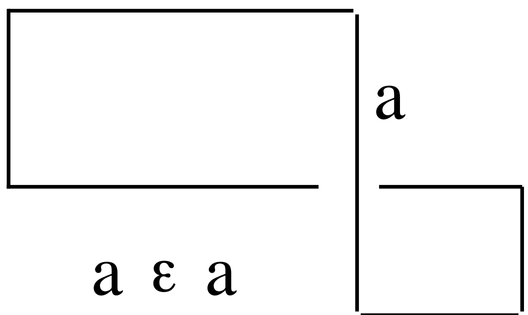


Knot Sets



$\boxed{a \, \varepsilon \, b}$

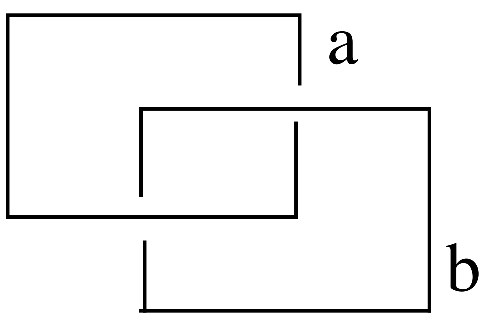
Crossing as Relationship



$a \ \varepsilon \ a$

$a = \{a\}$

Self-Membership

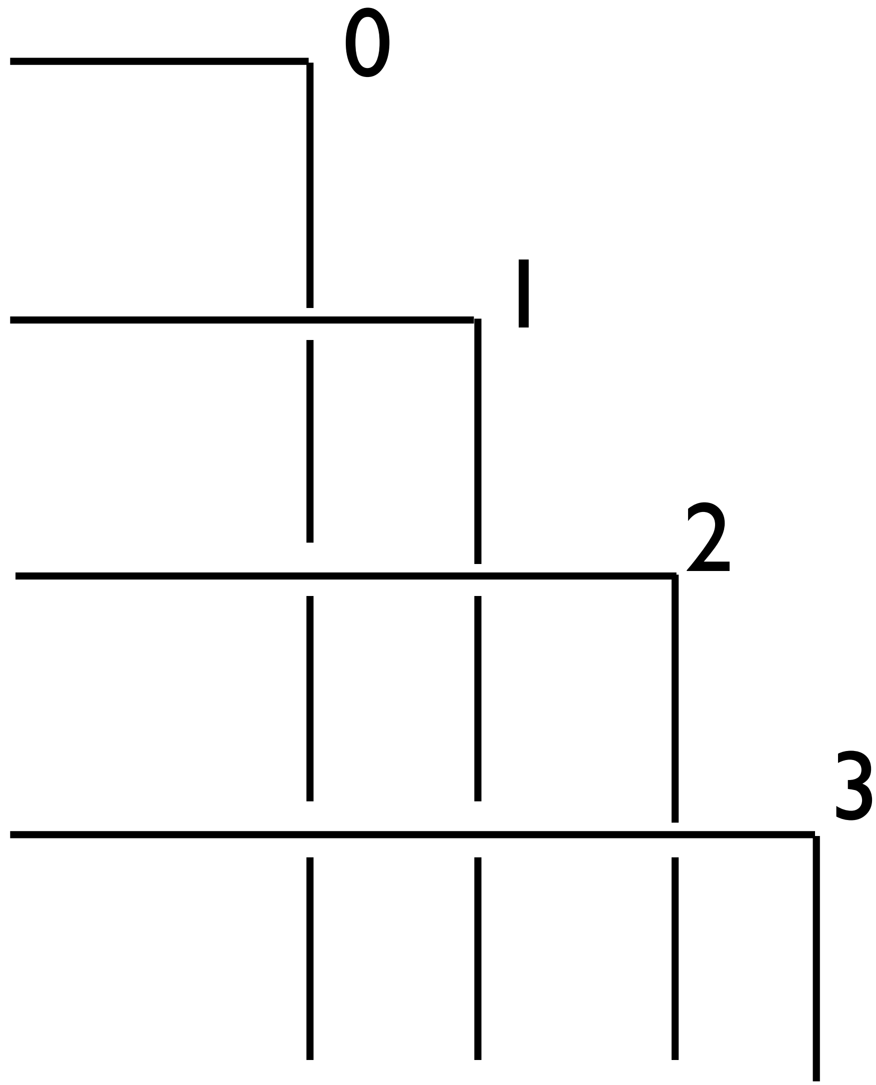


$a=\{b\}$

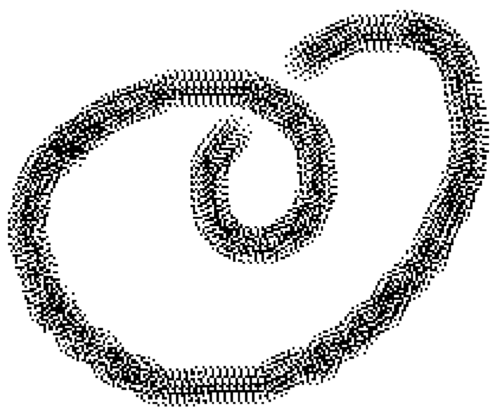
$b=\{a\}$

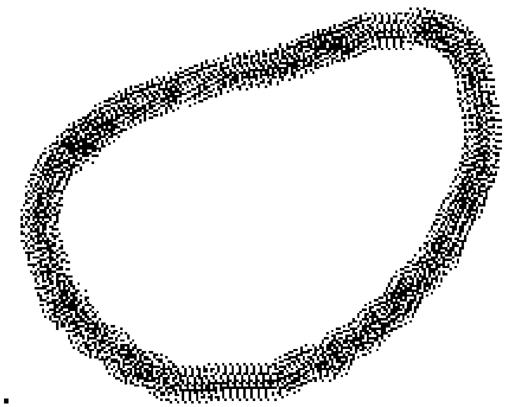
Mutuality

Architecture of Counting

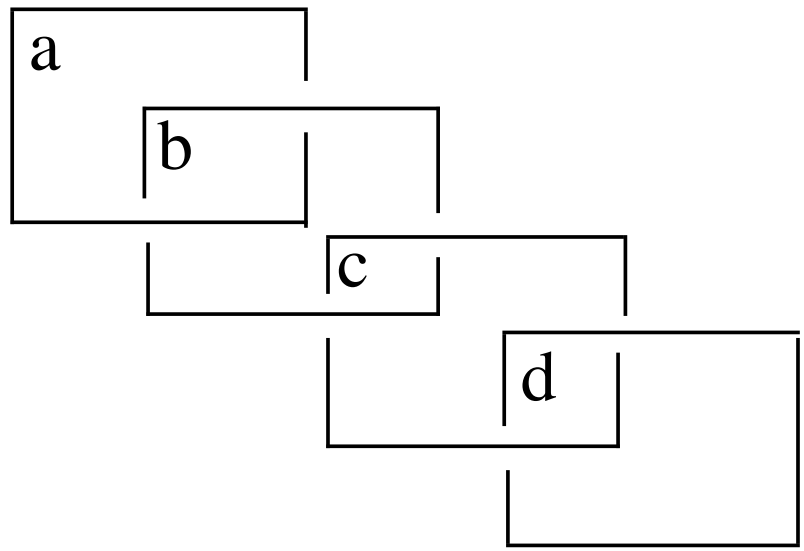


Topological Russell (K)not Paradox





A belongs to A. ⟺ A does not belong to A.

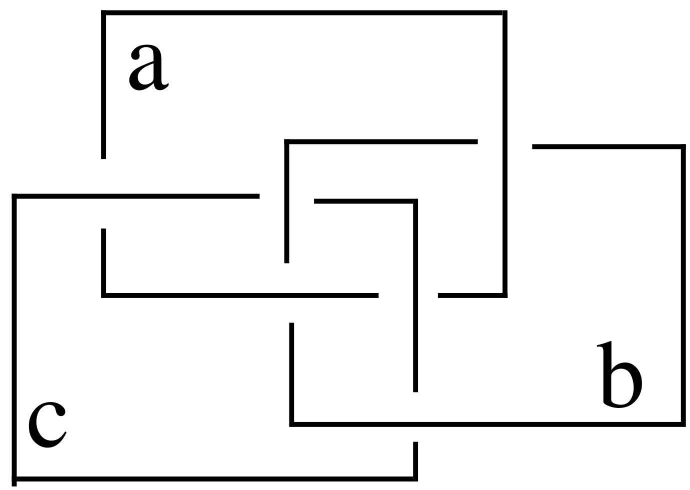


$a = \{b\}$

$b = \{a, c\}$

$c = \{b, d\}$

$d = \{c\}$

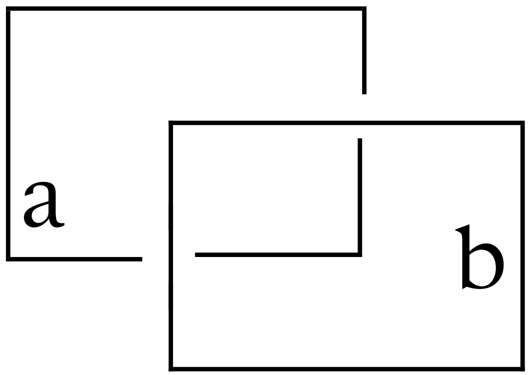


$a = \{b,b\}$

$b = \{c,c\}$

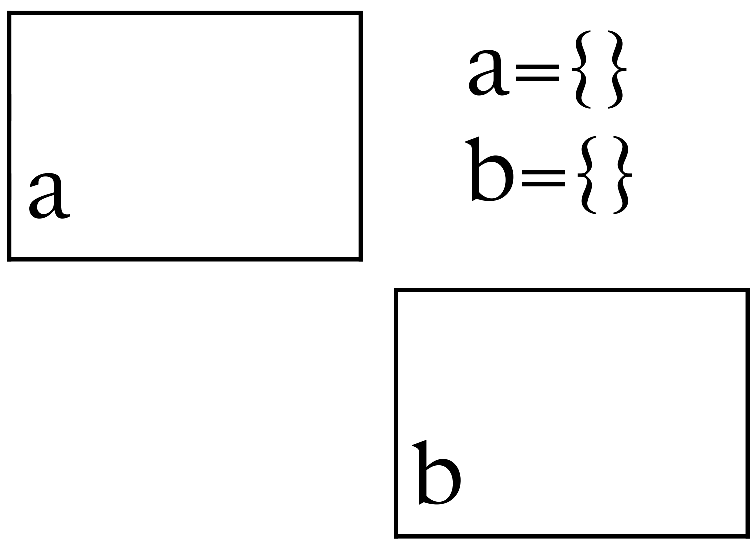
$c = \{a,a\}$

The Borrommean Rings



a = {}
b = {a,a}

topological
equivalence

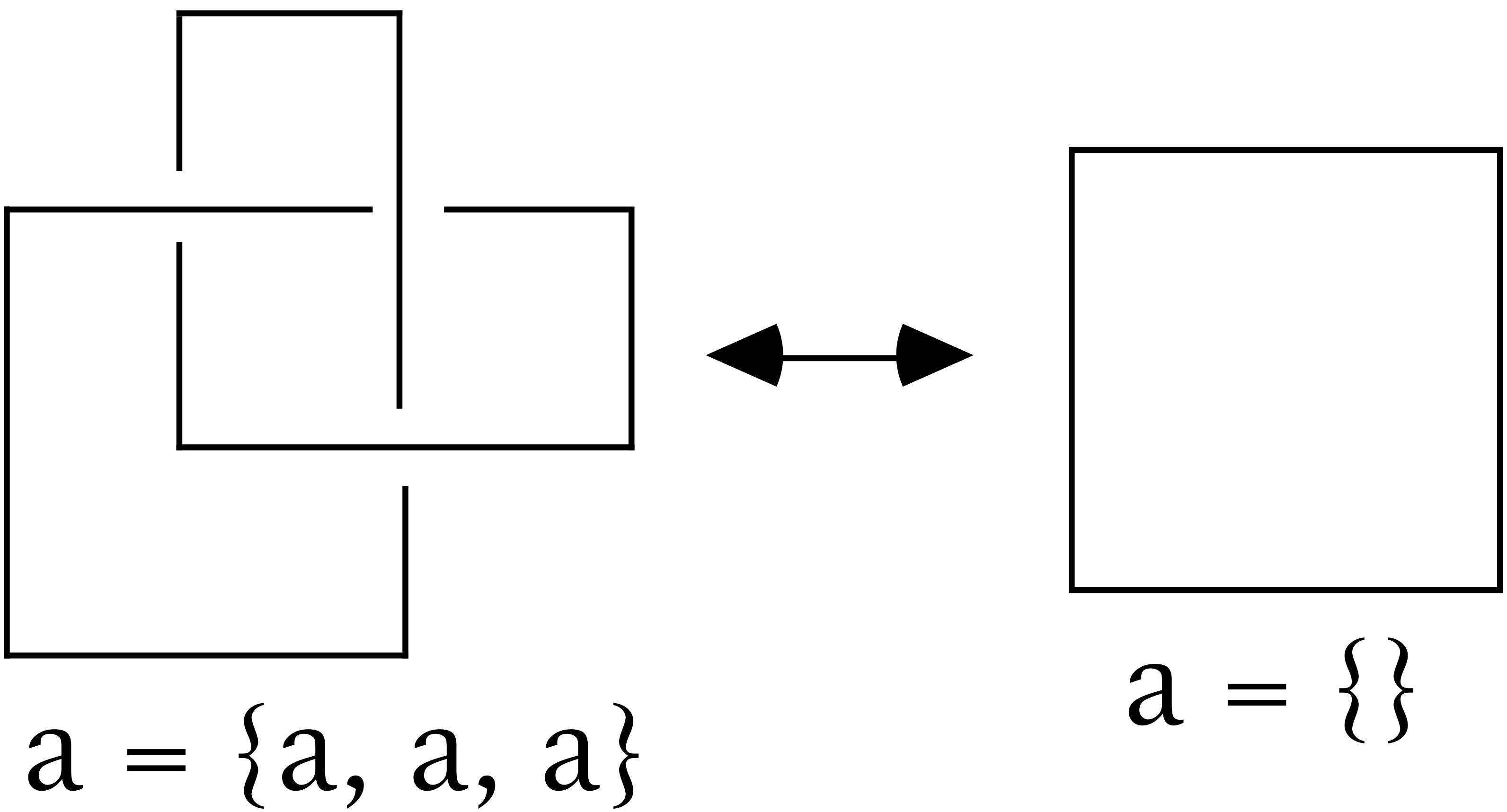


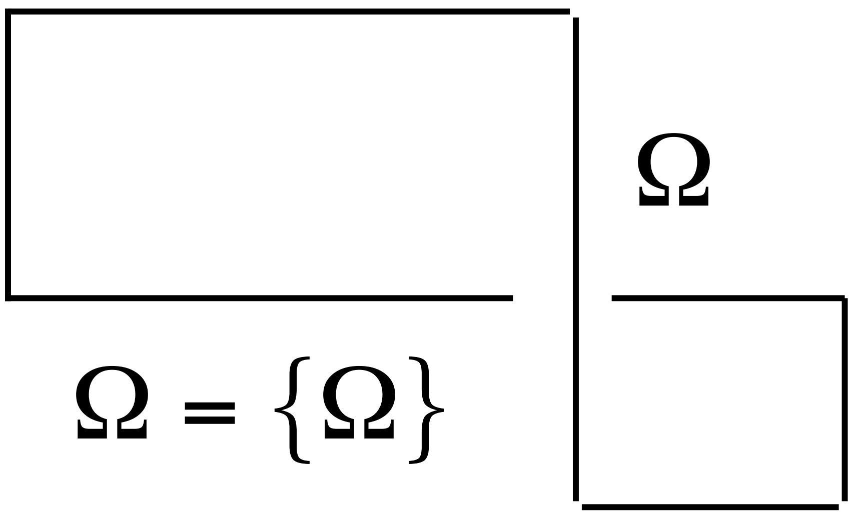
a={}
b={}

Knot Sets are "Fermionic".
Identical elements cancel in pairs.

(No problem with invariance under third Reidemeister move.)

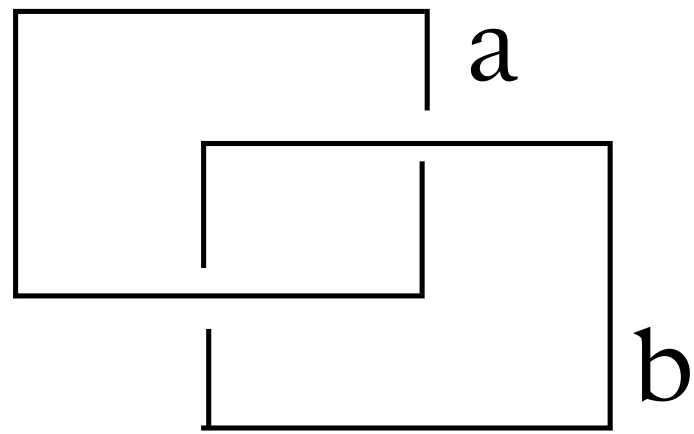
Alas, knot sets do not know knots. But they do provide a non-standard model for sets.





$\Omega = \{\Omega\}$

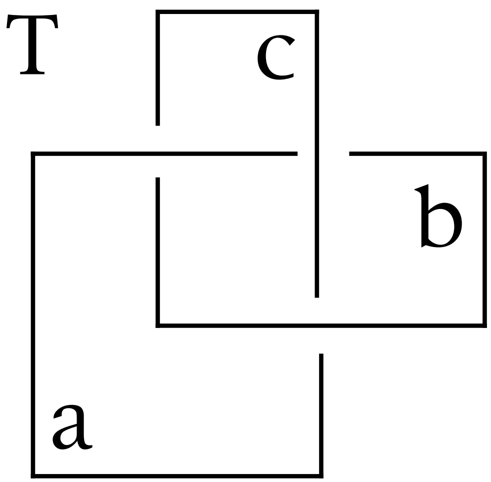
$\Omega \,\varepsilon\, \Omega$



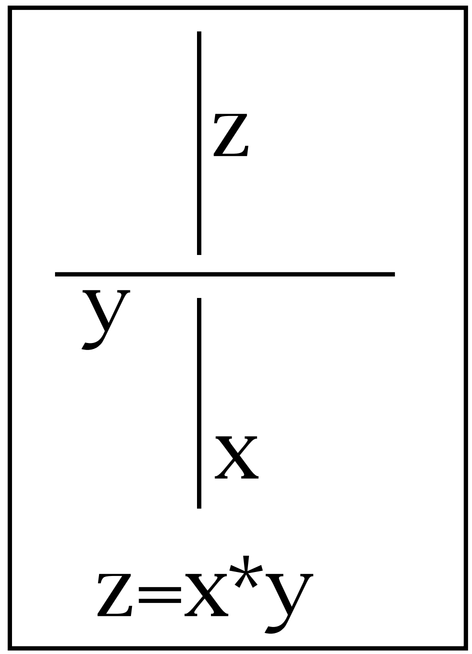
$a=\{b\}$

$b=\{a\}$

A Reflexive Algebra- The Quandle



$$b = a*c$$
$$c = b*a$$
$$a = c*b$$



$$a*a = a,$$
$$b*b = b,$$
$$c*c = c,$$
$$a*b = b*a = c,$$
$$b*c = c*b = a,$$
$$a*c = c*a = b.$$

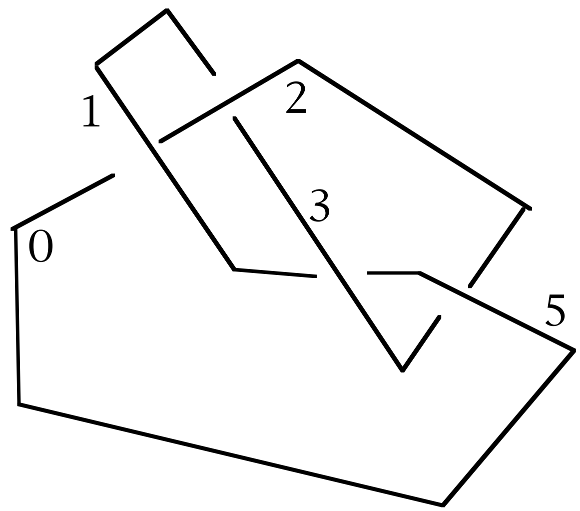
Non-Associative

$$(a*b)*c = c*c = c$$
$$a*(b*c) = a*a = a$$

Right-Distributive

$$(a*c)*(b*c) = b*a = c = (a*b)*c$$

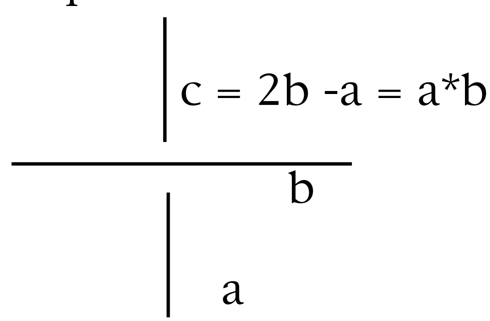
Here is the example for the Figure Eight Knot.



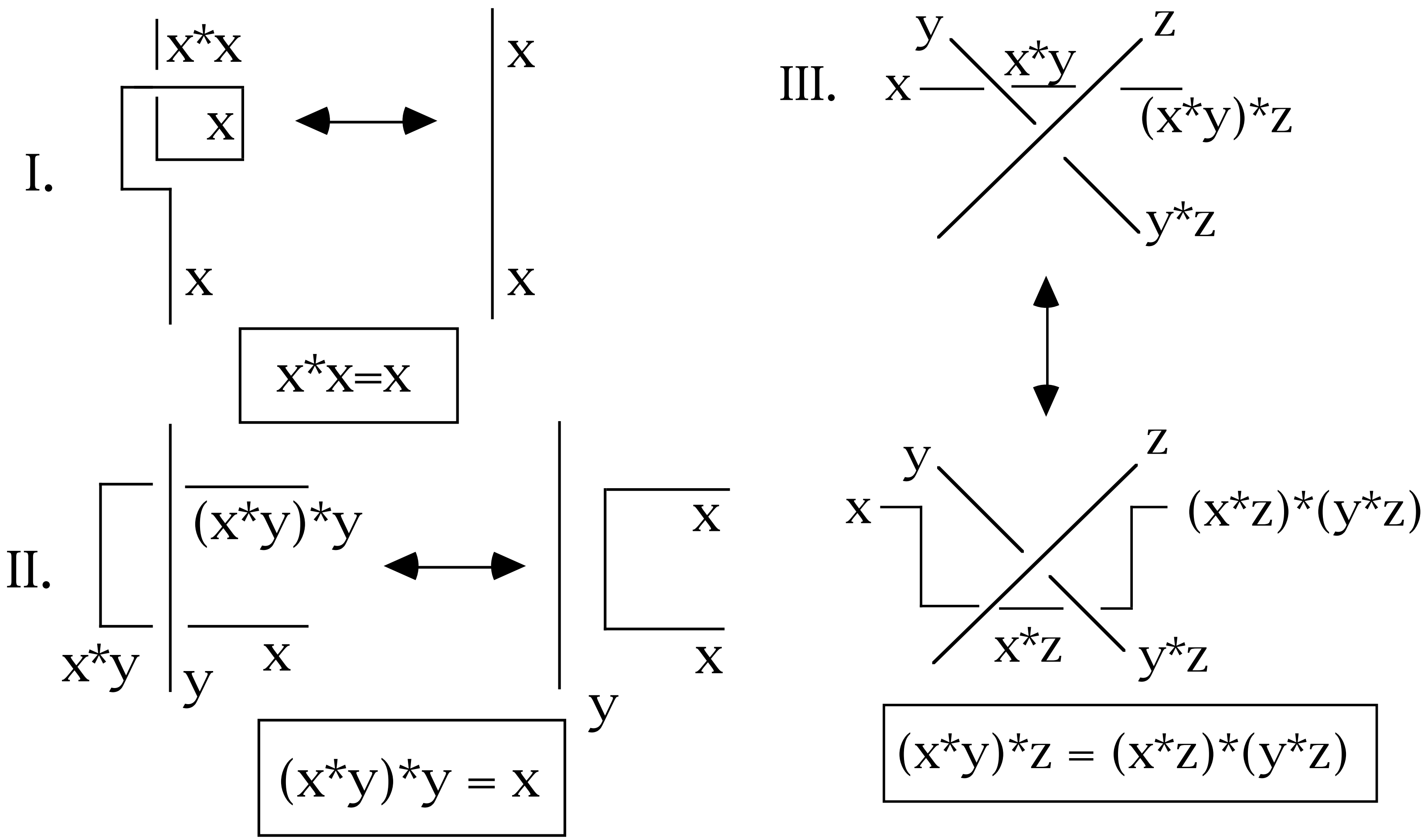
2 x 1 -0 = 1
2 x 2 -1 = 3
2 x 3 - 1 = 5
-> 0 = 5

Z/5Z = {0,1,2,3,4} with 0 = 5.

We have shown how an attempt to label the arcs of the knot according to the quandle rule



$$x*x = x$$
$$(x*y)*y = x$$
$$(x*y)*z = (x*z)*(y*z)$$



Left Distributivity

We have written the quandle as a right-distributive structure with invertible elements. It is mathematically equivalent to use the formalism of a left distributive operation. In left distributive formalism we have A*(b*c) = (A*b)*(A*c). This corresponds exactly to the interpretation that each element A in Q is a mapping of Q to Q where the mapping A[x] = A*x is a structure preserving mapping from Q to Q.

A[b*c] = A[b]*A[c].

We can ask of a domain that *every element of the domain is itself a structure preserving mapping of that domain.* This is very similar to the requirement of reflexivity and, as we have seen in the case of quandles, can often be realized for small structures such as the Trefoil quandle.

We call a domain M with an operation * that is left distributive a *magma*. Magmas are more general than the link diagrammatic quandles. We take only the analog of the third Reidemeister move and do not assume any other axioms. Even so there is much structure here. A magma with no other relations than left-distributivity is called a *free magma.*

Magma and Reflexivity

$$A*(B*C) = (A*B)*(B*C)$$

I shall call a magma M *reflexive* if it has the property that every structure preserving mapping of the algebra is realized by an element of the algebra *and* $(x*x)*z = x*z$ for all x and z in M.

Fixed Point Theorem for Reflexive Magmas. Let M be a reflexive magma. Let $F:M \dashrightarrow M$ be a structure preserving mapping of M to itself. Then there exists an element p in M such that $F(p) = p$.

Proof. Let $F:M \dashrightarrow M$ be any structure preserving mapping of the magma M to iteself. This means that we assume that $F(x*y) = F(x)*F(y)$ for all x and y in M. Define $G(x) = F(x*x)$ and regard $G:M \dashrightarrow M$. Is G structure preserving? We must compare
$G(x*y) = F((x*y)*(x*y)) = F(x*(y*y))$ with
$G(x)*G(y) = F(x*x)*F(y*y) = F((x*x)*(y*y))$.
Since $(x*x)*z = x*z$ for all x and z in M, we conclude that
$G(x*y) = G(x)*G(y)$ for all x and y in M.
Thus G is structure preserving and hence there is an element g of M such that $G(x) = g*x$ for all x in M. Therefore we have
$g*x = F(x*x)$, whence $g*g = F(g*g)$. For $p = g*g$, we have
$p = F(p)$. This completes the proof. //

This slide show has been only an introduction to certain mathematical and conceptual points of view about reflexivity. In the worlds of scientific, political and economic action these principles come into play in the way structures rise and fall in the play of realities that are created from (almost) nothing by the participants in their desire to profit, have power or even just to have clarity and understanding. Beneath the remarkable and unpredictable structures that arise from such interplay is a lambent simplicity to which we may return, as to the source of the world.